UNIT - I

MATRICES

Characteristic equation.

Let A be any square matrix of order n then the characteristic equation of A is $|A - \lambda I| = 0$

Method to find characteristic equation

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = sum of the main diagonal elements $a_{11} + a_{22}$

$$S_2 = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Hence the required characteristic equation for 3×3 matrix is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of the main diagonal elements $= a_{11} + a_{22} + a_{33}$

S_2 = sum of the minors of main diagonal elements

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

S_3 = Determinant value of $A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(1) Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Soln:

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The char. equation of A is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_1 = 1 + 2 = 3$

$S_2 = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Hence the required characteristic equation is $\lambda^2 - 3(\lambda) + 2 = 0$

$$\lambda^2 - 3\lambda + 2 = 0$$

2) Find the characteristic equation of $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2 + 1 + (-4) = 3 - 4 = -1$

$$S_2 = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$$

$= (-4-6) + (-8+5) + (2+9)$

$= -10 - 3 + 11$

$= -13 + 11$

$= -2$

$$S_3 = |A| = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$$

$= 2(-4-6) - (-3)(-12+15) + 1(6+5)$

$= 2(-10) + 3(3) + 1(11)$

$= -20 + 9 + 19$

$= 0$

Hence the required characteristic equation is

$\lambda^3 - (-1)\lambda^2 + (-2)\lambda + 0 = 0$

$\lambda^3 + \lambda^2 - 2\lambda = 0$ //

③ Find the characteristic polynomial of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

The ~~characteristic~~ equation $\lambda^2 - S_1\lambda + S_2$

$S_1 = 1 + 3 = 4$

$S_2 = |A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

Hence the characteristic polynomial is $\lambda^2 - 4\lambda - 5$.

Home work.

① Find the char. eqn of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ Ans:- $\lambda^2 - 4\lambda + 5 = 0$

② $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ Ans:- $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

EIGENVALUES AND EIGENVECTORS OF A REAL MATRIX

working rule to find Eigenvalues and Eigenvectors

Step 1:-

To find the characteristic equation $|A - \lambda I| = 0$

Step:-2

To solve the characteristic equation we get characteristic roots. They are called Eigenvalues.

Step:-3

To find Eigenvectors solve $(A - \lambda I)x = 0$ for the different values of λ.

I Problems based on Non-Symmetric matrices with non-Repeated Eigenvalues

① Find the Eigenvalues and Eigenvectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Step 1:-

To find the chara. eqn.

The characteristic equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where

$S_1 = 1 + 2 + 3 = 6$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0) = 4+5+2 = 11$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-2) - 0(\ ) + (-1)(2-4)$$

$$= 1(4) - 0 - 1(-2) = 4+2 = 6$$

$\therefore$ The char. eqn is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Step:-2

To find the roots of the chara. eqn.

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

If $\lambda = 1$ then $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root of $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda-1)(\lambda-3)(\lambda-2) = 0$$

$$\lambda = 1,\ \lambda = 2,\ \lambda = 3$$

Hence the eigenvalues of the given matrix 1, 2, 3

Step:-3

To find the Eigenvectors:-

To find the Eigenvectors solve $(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (A)}$$

Case(i) If $\lambda = 1$ then the equation(A) becomes

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_3 = 0$ —— (1)

$x_1 + x_2 + x_3 = 0$ —— (2)

$2x_1 + 2x_2 + 2x_3 = 0$ —— (3)

we choose here (1) and (2)

$0x_1 + 0x_2 - x_3 = 0$

$x_1 + x_2 + x_3 = 0$

Solving (1) & (2) we get

$$\frac{x_1}{0+1} = \frac{-x_2}{0+1} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

Hence a corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Case(2)

If $\lambda = 2$ then the equation (A) becomes

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_3 = 0$ —— (4)

$x_1 + x_3 = 0$ —— (5)

$2x_1 + 2x_2 + x_3 = 0$ —— (6)

we choose (5) & (6)

$x_1 + 0x_2 + x_3 = 0$

$2x_1 + 2x_2 + x_3 = 0$

$$\frac{x_1}{0-2} = \frac{-x_2}{1-2} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

Hence the corresponding Eigenvector $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Case (3)

If $\lambda = 3$ then the equation (A) becomes

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0x_2 - x_3 = 0 \quad \text{—— (7)}$$

$$x_1 - x_2 + x_3 = 0 \quad \text{—— (8)}$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \text{—— (9)}$$

Solving (8) and (9) we get

$$\frac{x_1}{0-2} = \frac{-x_2}{0-2} + \frac{x_3}{2+2}$$

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

Hence the corresponding Eigenvector $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Result:

1. Eigenvalues of the given matrix A are 1, 2, 3
2. If $\lambda = 1$ then the Corresponding Eigenvector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
3. If $\lambda = 2$ " " " $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$
4. If $\lambda = 3$ then the corresponding Eigenvector $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Home work

1. Find the Eigenvalues and Eigenvectors of

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Ans:- Eigenvalues $-1, 1, 2$

Eigenvector $x_1 = [1, 0, 1]$ $x_2 = [3, 2, 1]$ $\cdot$ $x_3 = [1 \ 3 \ 1]$

2. $$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Ans:- Eigen values $-2, 1, 3$

Eigenvector $x_1 = [11, 1, -14]$, $x_2 = [-1, 1, 1]$ $x_3 = [1, 1, 1]$

II problems based on Non-Symmetric matrix with Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Soln: Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Step 1 :

To find the characteristic equation

The charac. eqn of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements $= (-2) + (1) + (0) = -1$

$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= (0-12) + (0-3) + (-2-4)$

$= -12 - 3 - 6$

$= -21$

$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= (-2)(0-12) - 2(0-6) + (-3)(-4+1)$

$= 24 + 12 + 9$

$= 45$

$\therefore \lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

Step:-2

To solve the characteristic equation

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

If $\lambda = 1$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 1 + 1 - 21 - 45 \neq 0$

If $\lambda = -1$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = -1 + 1 + 21 - 45 \neq 0$

If $\lambda = 2$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 8 + 4 - 42 - 45 \neq 0$

If $\lambda = -2$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = -8 + 4 + 42 - 45 \neq 0$

If $\lambda = 3$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 27 + 9 - 63 - 45 \neq 0$

If $\lambda = -3$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = -27 + 9 + 63 - 45 = 0$

Hence $\lambda = -3$ is a root of $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\therefore \quad \lambda^3+\lambda^2-21\lambda-45 = (\lambda+3)(\lambda^2-2\lambda-15)=0$

$\lambda+3=0 \qquad \lambda^2-2\lambda-15=0$

$\lambda=-3 \qquad (\lambda-5)(\lambda+3)=0$

$\lambda=-3, \quad \lambda=5, \quad \lambda=-3$

$\therefore$ the Eigenvalues are $-3, -3, 5$

<u>Step:-3</u>

To find Eigenvectors

To find the Eigenvectors solve $(A-\lambda I)X=0$

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (A)}$$

<u>Case(i)</u>

If $\lambda=-3$ then the eqn (A) becomes

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+2x_2-3x_3=0$ --- (1)

$2x_1+4x_2-6x_3=0$ --- (2)

$-x_1-2x_2+3x_3=0$ --- (3)

Here (1) (2) and (3) are Same equations

$x_1+2x_2-3x_3=0$

put $x_1=0$ we get $2x_2=3x_3$

$$\frac{x_2}{3}=\frac{x_3}{2}$$

Hence the corresponding Eigenvector is $X_1=\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

put $x_2=0$ we get $x_1-3x_3=0$

$x_1=3x_3$

$$\frac{x_1}{3}=\frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_2=\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Case (2)

If $\lambda = 5$ then the equation (A) becomes

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \text{--- (4)}$$

$$2x_1 - 4x_2 - 6x_3 = 0 \quad \text{--- (5)}$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \text{--- (6)}$$

Solving (4) and (5) we get

$$\frac{x_1}{-12-12} = \frac{-x_2}{42+6} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence the corresponding Eigenvector $X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Result:-

1. Eigenvalues of the given matrix A are $-3, -3, 5$
2. If $\lambda = -3$ then the corresponding Eigenvector is $X_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$
   If $\lambda = -3$ then the corresponding Eigenvector is $X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$
3. If $\lambda = 5$ then the corresponding Eigenvector is $X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Home work:-

① Find the Eigenvalue and Eigenvectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Ans :- Eigenvalue $1, 1, 5$
Eigenvector $X_1 = [2, -1, 0]$ $X_2 = [1, 2, -5]$ $X_3 = [1, 1, 1]$

② $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ Ans:- Eigenvalue $-1, -1, 3$
Eigenvector $X_1 = [0, 1, -1]$; $X_2 = [1, 0, 2]$; $X_3 = [1, 1, 2]$

III problems based on Symmetric matrices with Non-Repeated Eigenvalues

① Find the Eigenvalues and Eigenvectors of the matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Step:- 1

To find the characteristic equation

The characteristic equation of the given matrix is $|A-\lambda I|=0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 7+6+5 = 18$$

$$S_2 = \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$$= (30-4)+(35-0)+(42-4)$$

$$= 26+35+38 = 99$$

$$S_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7(30-4)+2(-10-0)+0(\quad)$$

$$= 7(26)-20$$

$$= 182-20 = 162$$

The characteristic equation of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Step:-2

To solve the characteristic equation

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

If $\lambda=1$ then $1-18+99-162 \neq 0$

If $\lambda=-1$ then $-1-18-99-162 \neq 0$

If $\lambda=2$ then $8-72+198-162 \neq 0$

If $\lambda=-2$ then $-8-72-198-162 \neq 0$

If $\lambda=3$ then $27-162+297-162=0$

$\therefore \lambda=3$ is a root

3	1	-18	99	-162
	0	3	-45	162
	1	-15	54	0

$$\therefore \lambda^3 - 18\lambda^2 + 99\lambda - 162 = (\lambda-3)(\lambda^2-15\lambda+54)=0$$

$$\lambda - 3 = 0 \qquad \lambda^2 - 15\lambda + 54 = 0$$

Scanned by CamScanner

$$\lambda^2 - 9\lambda - 6\lambda + 54 = 0$$

$$\lambda(\lambda-9) - 6(\lambda-9) = 0$$

$$(\lambda-9)(\lambda-6) = 0$$

$$\lambda = 6, 9$$

Hence the Eigenvalues are 3, 9, 6

Step:-3

To get the eigenvectors solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Case(i) when $\lambda = 3$ we get

$$4x_1 - 2x_2 + 0x_3 = 0 \quad \text{--- (1)}$$

$$-2x_1 + 3x_2 - 2x_3 = 0 \quad \text{--- (2)}$$

$$0x_1 - 2x_2 + 2x_3 = 0 \quad \text{--- (3)}$$

Solving (2) & (3) we get

$$\frac{x_1}{6-4} = \frac{-x_2}{-4-0} = \frac{x_3}{4-0}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence the Corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case(ii) when $\lambda = 6$ we get

$$x_1 - 2x_2 + 0x_3 = 0 \quad \text{--- (1)}$$

$$-2x_1 + 0x_2 - 2x_3 = 0 \quad \text{--- (2)}$$

$$0x_1 - 2x_2 - x_3 = 0 \quad \text{--- (3)}$$

Solving (2) & (3) we get

$$\frac{x_1}{0-4} = \frac{-x_2}{2-0} = \frac{x_3}{4-0}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the Corresponding Eigenvector is $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case(iii) When $\lambda = 9$ we get

$-2x_1 - 2x_2 + 0x_3 = 0$ ——— ①

$-2x_1 - 3x_2 - 2x_3 = 0$ ——— ②

$0x_1 - 2x_2 - 4x_3 = 0$ ——— ③

Solving (2) & (3) we get

$$\frac{x_1}{12-4} = \frac{-x_2}{8-0} = \frac{x_3}{4-0}$$

$$\frac{x_1}{8} = \frac{x_2}{-8} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence the Corresponding Eigenvector is $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Result:-

1. Eigenvalues of the given matrix of A are 3, 6, 9
2. The corresponding Eigenvectors of A are

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Home work:-

① $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$ Eigenvalues $-3, -6, 12$

Eigenvectors $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

② $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ Eigenvalues $-2, 4, 6$

Eigenvectors $X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

PROBLEMS BASED ON SYMMETRIC MATRICS WITH REPEATED EIGENVALUES

① Find the Eigenvalues and Eigenvectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Soln:-

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Step:-1

To find the Characteristic equation

The Characteristic equation of the given matrix is $|A-\lambda I|=0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements

$= 0+0+0 = 0$

$$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= (0-1)+(0-1)+(0-1)$

$= -3$

$S_3 = |A|$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0(0-1) - 1(0-1) + 1(1-0)$$

$= 0+1+1$

$= 2$

Therefore the characteristic equation is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

(i.e) $\lambda^3 - 3\lambda - 2 = 0$

Step:-2

To Solve the characteristic equation $\lambda^3 - 3\lambda - 2 = 0$

If $\lambda = 1$ then $\lambda^3 - 3\lambda - 2 = 1-3-2 \neq 0$

$\lambda = -1$ then $\lambda^3 - 3\lambda - 2 = -1+3-2 = 0$

$\therefore \lambda = -1$ is a root

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda+1)(\lambda-2) = 0$

Hence the Eigenvectors are $-1, -1, 2$

Step:-3

To find the Eigenvectors

Scanned by CamScanner

To find the Eigenvectors solve $(A-\lambda I)x=0$

$$\left[\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{—— (A)}$$

Case (i)

If $\lambda = 2$ then the equation (A) becomes

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0 \quad \text{—— (1)}$$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{—— (2)}$$

$$x_1 + x_2 - 2x_3 = 0 \quad \text{—— (3)}$$

Solving (1) and (2)

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence the Corresponding Eigenvector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (ii)

If $\lambda = -1$ then the equation (A) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0 \quad \text{—— (4)}$$

$$x_1 + x_2 + x_3 = 0 \quad \text{—— (5)}$$

$$x_1 + x_2 + x_3 = 0 \quad \text{—— (6)}$$

Hence (4) (5) (6) represents the same equation

$$x_1 + x_2 + x_3 = 0$$

put $x_1 = 0$ we get $x_2 = -x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

Hence the Corresponding Eigenvector $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as x_3 is orthogonal to x_1 and x_2

Since the given matrix is Symmetric

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad (\text{or}) \quad l + m + n = 0 \quad \text{——} (7)$$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad (\text{or}) \quad 0l + m - n = 0 \quad \text{——} (8)$$

Solving (7) and (8) we get

$$\frac{l}{-1-1} = \frac{-m}{-1-0} = \frac{n}{1-0}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$$

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{-1}$$

Hence the Corresponding Eigenvector $X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Result:-

1. Eigenvalues of the given matrix A are $2, -1, -1$
2. Eigenvectors Corresponding to the Eigenvalues are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Home work:-

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ Eigen values $0, 0, 14$

Eigenvector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ Eigenvalues $1, 3, 3$

Eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Properties of Eigenvalues and Eigenvectors

Property 1

The sum of the Eigenvalues of a matrix is the sum of the elements of the principal diagonal.

proof:-

let A be a square matrix of order n.

The characteristic equation of A is $|A-\lambda I| = 0$

(i.e.) $\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \dots + (-1)^n S_n = 0$ —— ①

where

S_1 = sum of the diagonal elements of A.

. . . .

. . . . . .

S_n = determinat of A

we know the roots of the characteristic equations are called Eigenvalues of the given matrix

solving (1) we get n roots.

let the n roots be $\lambda_1, \lambda_2, \dots \lambda_n$.

(i.e.) $\lambda_1, \lambda_2, \dots \lambda_n$ are the Eigenvalues of A.

we know already

λ^n − (sum of the roots) λ^{n-1} + [sum of the roots taken two at a time] $\lambda^{n-2} - \dots + (-1)^n$ (product of the roots) $= 0$ —— ②

sum of the roots $= S_1$ by (1) & (2)

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = S_1$$

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + \dots + a_{nn}$$

$\boxed{\text{Sum of the Eigenvalues = sum of the main diagonal elements}}$

product of the roots $= S_n$ by (1) & (2)

$$\lambda_1\lambda_2 \dots \lambda_n = \text{det of } A$$

$\boxed{\text{product of the Eigenvalues} = |A|}$

PROPERTY:-2

A square matrix A and its transpose A^T have the same Eigenvalues.

proof:-

let A be a square matrix of order n.

The characteristic equation of A and A^T are

$|A-\lambda I| = 0$ —① and $|A^T - \lambda I| = 0$ —②

Since the determinant value is unaltered by the interchanged of rows and columns

we know $|A| = |A^T|$ Hence (1) and (2) are identical.

∴ The Eigenvalues of A and A^T are the same.

Property :-3

The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Proof:-

Let us consider the triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

on expansion it gives $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$

$$\lambda = a_{11}, a_{22}, a_{33}$$

which are diagonal elements of matrix A.

Property : 4

If λ is an Eigenvalue of a matrix A, then $1/\lambda$ is the Eigenvalue of A^{-1}.

Proof!-

If X be the Eigenvector corresponding to λ, then $AX = \lambda X$ —①

premultiplying both sides by A^{-1}, we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

(ie) $A^{-1}X = \frac{1}{\lambda}X$

This being of the same form as (i) shows that $1/\lambda$ is an Eigenvalue of the inverse matrix A^{-1}.

Property 5:

If λ is an Eigenvalues of an orthogonal matrix then $1/\lambda$ is also its Eigenvalue.

Proof:

Defn: orthogonal matrix

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

(ie) $A^T = A^{-1}$

Let A be an orthogonal matrix. Given λ is an Eigenvalue of A.

$\Rightarrow$ $1/\lambda$ is an Eigenvalue of A^{-1}. Since $A^T = A^{-1}$

$\therefore$ $1/\lambda$ is an Eigenvalue of A^T Since But the matrices A and A^T have the same Eigenvalues, since the determinants $|A-\lambda I|$ and $|A^T-\lambda I|$ are the same.

Hence $1/\lambda$ is also an Eigenvalue of A.

Property 6:

If $\lambda_1, \lambda_2 \dots \lambda_n$ are the Eigenvalues of a matrix A, then A^m has the Eigenvalues $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$.

Proof:

Let λ_i be the Eigenvalues of A and x_i the corresponding Eigenvector.

Then $Ax_i = \lambda_i x_i$ —①

we have $A^2 x_i = A(Ax_i)$

$= A(\lambda_i x_i)$

$= \lambda_i A(x_i)$

$= \lambda_i (\lambda_i x_i)$

$= \lambda_i^2 x_i$

III^{ly}

$A^3 x_i = \lambda_i^3 x_i$

In general $A^m x_i = \lambda_i^m x_i$ —②

(1) and (2) are in same shape.

Hence λ_i^m is an Eigenvalue of A^m. The corresponding Eigenvector is the same x_i

Property :- 8

The Eigenvectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.

Proof:
For a real symmetric matrix A, the Eigenvalues are real.
Let x_1, x_2 be Eigenvectors corresponding to two distinct eigen values λ_1, λ_2 [λ_1, λ_2 are real]

$$Ax_1 = \lambda_1 x_1 \quad \text{---(1)}$$

$$Ax_2 = \lambda_2 x_2 \quad \text{---(2)}$$

Premultiplying (1) by x_2^T we get

$$x_2^T A x_1 = x_2^T \lambda_1 x_1$$

$$= \lambda_1 x_2^T x_1$$

||| ly

$$x_1^T A x_2 = \lambda_2 x_1^T x_2 \quad \text{---(3)}$$

But $(x_2^T A x_1)^T = (\lambda_1 x_2^T x_1)^T$

$$x_1^T A^T x_2 = \lambda_1 x_1^T x_2$$

$$x_1^T A x_2 = \lambda_1 x_1^T x_2 \quad \text{---④} \quad [\because A^T = A] \text{ since symmetric matrix}$$

From (3) and (4)

$$\lambda_1 x_1^T x_2 = \lambda_2 x_1^T x_2$$

$$(\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$$\lambda_1 \neq \lambda_2, \quad x_1^T x_2 = 0$$

$\therefore x_1, x_2$ are orthogonal.

Property :- 9

The similar matrices have same Eigenvalues.

Proof:
Let A, B be two similar matrices
Then there exists a non-singular matrix P such that $B = P^{-1}AP$

$$B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - P^{-1}\lambda I P$$

$$= P^{-1}(A-\lambda I)P$$

$$|B-\lambda I| = |P^{-1}|\ |A-\lambda I|\ |P|$$

$$= |A-\lambda I|\ |P^{-1}P|$$

$$= |A-\lambda I|\ |I|$$

$$= |A-\lambda I|$$

Therefore A, B have the same characteristic function and hence characteristic roots.

$\therefore$ They have same Eigenvalues.

Problems based on Properties

① Find the sum and product of the Eigenvalues of the matrix.

(i) $A = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & -4 & 4 \\ 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Soln:

(i) Sum of the Eigenvalues = sum of its diagonal elts $= 2-2=0$

Product of the Eigenvalues $= |A| = \begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} = -4+12 = 8$

(ii) Sum of the Eigenvalues = sum of its diagonal elts $= 1-2+3=2$

Product of the Eigenvalues $= |A| = \begin{vmatrix} 1 & -4 & 4 \\ 1 & -2 & 4 \\ 2 & -1 & 3 \end{vmatrix}$

$$= 1(-6+4) + 4(3-8) + 4(-1+4)$$

$$= 1(-2) + 4(-5) + 4(3) = -2-20+12 = -10$$

(iii) Sum of the Eigenvalues = sum of its diagonal elts $= 1+2+1=4$

Product of Eigenvalues $= |A| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$$= 1(2-1) - 2(-1-1) + 3(-1-2)$$

$$= 1(1) - 2(-2) + 3(-3)$$

$$= 1+4-9 = -4.$$

② If 2, 2, 3 are the Eigenvalues of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ Find the Eigenvalues of A^T.

Soln:

A Square matrix A and its transpose A^T have the same Eigenvalues.

Hence Eigenvalues of A^T are 2, 2, 3.

③ Two of the Eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6 Find the Eigenvalues of A^{-1}.

Soln:

Sum of the Eigenvalues = Sum of the main diagonal elements

$= 3+5+3 = 11$

Let K be the third Eigenvalue

$\therefore 3+6+K = 11$

$9+K = 11$

$\boxed{K = 2}$

$\therefore$ The Eigenvalues of A^{-1} are $1/2, 1/3, 1/6$.

④ Find the Eigenvalues of A^3 given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Soln:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

clearly given A is a upper triangular matrix

Hence the Eigenvalues are 1, 2, 3.

The Eigenvalues of the given matrix A are 1, 2, 3

By property the Eigenvalues of the matrix A^3 are $1^3, 2^3, 3^3$

property: If $\lambda_1, \lambda_2 \dots \lambda_n$ are Eigenvalues of A then $\lambda_1^k, \lambda_2^k, \dots \lambda_n^k$ are Eigenvalues of A^k for any positive integer.

⑤ The Eigenvectors of a 3×3 real Symmetric matrix A corresponding to the Eigenvalues 2, 3, 6 are $[1, 0, -1]^T$, $[1, 1, 1]^T$ and $[-1, 2, -1]^T$ respectively, find the matrix A.

Soln: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be the matrix

Cha. eqn is $(\lambda-2)(\lambda-3)(\lambda-6)=0$

$$(\lambda-2)\left[\lambda^2-6\lambda-3\lambda+18\right]=0$$

$$(\lambda-2)\left[\lambda^2-9\lambda+18\right]=0$$

$$\lambda^3-9\lambda^2+18\lambda-2\lambda^2+18\lambda-36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

when $\lambda=2$

$$\begin{bmatrix} a_1-2 & b_1 & c_1 \\ a_2 & b_2-2 & c_2 \\ a_3 & b_3 & c_3-2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=0$$

$$(a_1-2)-c_1=0$$

$$a_2-c_2=0$$

$$a_3-(c_3-2)=0$$

$$a_1-c_1=2 \quad \text{—— (1)}$$

$$a_2-c_2=0 \quad \text{—— (2)}$$

$$a_3-c_3=-2 \quad \text{—— (3)}$$

when $\lambda=3$

$$\begin{pmatrix} a_1-3 & b_1 & c_1 \\ a_2 & b_2-3 & c_2 \\ a_3 & b_3 & c_3-3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=0$$

$$(a_1-3)+b_1+c_1=0$$

$$a_2+(b_2-3)+c_2=0$$

$$a_3+b_3+(c_3-3)=0$$

$$a_1+b_1+c_1=3 \quad \text{—— (4)}$$

$$a_2+b_2+c_2=3 \quad \text{—— (5)}$$

$$a_3+b_3+c_3=3 \quad \text{—— (6)}$$

when $\lambda = 6$

$$\begin{pmatrix} a_1-6 & b_1 & c_1 \\ a_2 & b_2-6 & c_2 \\ a_3 & b_3 & c_3-6 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$-(a_1-6)+2b_1-c_1=0$$
$$-a_2+2(b_2-6)-c_2=0$$
$$-a_3+2b_3-(c_3-6)=0$$

(ie) $-a_1+6+2b_1-c_1=0$

$-a_2+2b_2-12-c_2=0$

$-a_3+2b_3-c_3+6=0$

$$a_1-2b_1+c_1=6 \quad \text{—(7)}$$
$$a_2-2b_2+c_2=-12 \quad \text{—(8)}$$
$$a_3-2b_3+c_3=6 \quad \text{—(9)}$$

Solving (1), (4) and (7)

$$a_1-c_1=2 \quad \text{—(1)}$$
$$a_1+b_1+c_1=3 \quad \text{—(4)}$$
$$a_1-2b_1+c_1=6 \quad \text{—(7)}$$

(4) - (7) $\Rightarrow$ $3b_1=-3$

$$\boxed{b_1=-1}$$

(4) $\Rightarrow$ $a_1+c_1=4$ —(10)

(1) + (10) $\Rightarrow$ $2a_1=6$

$$\boxed{a_1=3}$$

(1) $\Rightarrow$ $3-c_1=2$

$3-2=c_1$

$$\boxed{c_1=1}$$

$a_1=3,\quad b_1=-1 \quad c_1=1$

Solving (2) (5) and (8)

$$a_2-c_2=0 \quad \text{—(2)}$$
$$a_2+b_2+c_2=3 \quad \text{—(5)}$$
$$a_2-2b_2+c_2=-12 \quad \text{—(8)}$$

(5) - (8) $\Rightarrow$ $3b_2=15$

$$\boxed{b_2=5}$$

$(8) \Rightarrow a_2 + c_2 = -2$ —(11)

$(2) + (11) \Rightarrow 2a_2 = -2$

$\boxed{a_2 = -1}$

$(2) \Rightarrow -1 - c_2 = 0$

$\boxed{c_2 = -1}$

$\therefore a_2 = -1, \; b_2 = 5, \; c_2 = -1$

Solving (3), (6) and (9)

$a_3 - c_3 = -2$ —(3)

$a_3 + b_3 + c_3 = 3$ —(6)

$a_3 - 2b_3 + c_3 = 6$ —(9)

$(6) - (9) \Rightarrow 3b_3 = -3$

$\boxed{b_3 = -1}$

$(6) \Rightarrow a_3 + c_3 = 4$ —(12)

$(3) + (12) \Rightarrow 2a_3 = 2$

$\boxed{a_3 = 1}$

$(3) \Rightarrow 1 - c_3 = -2$

$\boxed{c_3 = 3}$

$\therefore a_3 = 1 \quad b_3 = -1 \quad c_3 = 3$

$$\therefore A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Verification:-

Sum of the Eigenvalues = Sum of the main diagonal elts.

$2 + 3 + 6 = 3 + 5 + 3$

$2 + 3 + 6 = 11$

Product of the Eigenvalues = $|A|$

$$(2)\,(3)\,(6) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$2[18] = 3(15-1) + 1(-3+1) + 1(1-5)$

$= 3(14) - 2 - 4 = 42 - 6 = 36$

CAYLEY - HAMILTON THEOREM

Statement:- Every square matrix satisfies its own characteristic equation

uses of Cayley Hamilton Theorem

(i) the positive integral powers and
(ii) the inverse of a square matrix A.

problems:-

① Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation.

Soln: Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The Cha. eqn of the given matrix is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = sum of the main diagonal elements

$= 1 + 1 = 2$

$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$

∴ The cha. equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore the given matrix satisfies its own characteristic equation.

② Using Cayley - Hamilton theorem find A^4 when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln: S_1 = sum of its leading diagonals $= 2 + 2 + 2 = 6$

S_2 = sum of the minors of its leading diagonals

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1)$$

$$= 3+2+3 = 8$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 2(3) + 1(-1) + 2(-1)$$

$$= 6-1-2 = 3$$

$\therefore$ The characteristic equation of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley - Hamilton theorem

$$A^3 - 6A^2 + 8A - 3I = 0$$

To find A^4

multiply A on both sides

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6[6A^2 - 8A + 3I] - 8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I$$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

③ Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ using Cayley - Hamilton theorem.

Soln:

The char. equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = 1 + 2 - 1 = 2$

$S_2 = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$

$= (-2+1) + (-1-8) + (2+3)$

$= -1 - 9 + 5 = -5$

$S_3 = |A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$

$= 1(-2+1) + 1(-3+2) + 4(3-4)$

$= 1(-1) + 1(-1) + 4(-1)$

$= -1 - 1 - 4 = -6$

∴ The Char. equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

by Cayley Hamilton Theorem we get

$A^3 - 2A^2 - 5A + 6I = 0$

To find A^{-1}

$\div$ by A $\Rightarrow A^2 - 2A - 5I + \frac{6}{A} = 0$

$A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I$

$A^{-1} = \frac{1}{6}\left[-A^2 + 2A + 5I\right]$

$A^2 = A \times A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$

$$A^{-1} = \frac{1}{6}\left[-A^2 + 2A + 5I\right]$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

④ Use Cayley-Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Soln:

The char. eqn. of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2 + 1 + 2 = 5$$

$$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0) = 2 + 3 + 2 = 7$$

$$S_3 = |A|$$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$

$$= 4 - 1$$

$$= 3$$

$\therefore$ the Char. eqn is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

by Cayley-Hamilton th. we get

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{——} \quad (1)$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3) + A(A^3 - 5A^2 + 8A - 2) + I$$

$$= A^5(0) + A[(A^3 - 5A^2 + 7A - 3] + [A + 1 + I]$$

$$= 0 + A\{[0] + A + 1 + I\}$$

$$= A^2 + A + I \quad \text{—(2)}$$

Now $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

(2) => Given value $= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Home work:-

① verify Cayley Hamilton theorem and find its inverse

(i) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ Ans:- $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ Ans:- $A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

② If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ prove that $A^3 - 3A^2 - 9A - 5I = 0$

Hence find A^4 and A^{-1} Ans:- $A^4 = \begin{bmatrix} 209 & 208 & 208 \\ 208 & 209 & 208 \\ 208 & 208 & 209 \end{bmatrix}$

SIMILARITY TRANSFORMATION - ORTHOGONAL TRANSFORMATION

DIAGONALISATION

Working rule for Diagonalisation

Step 1:- To find the char. equation

Step 2:- To solve the char. equation

Step 3:- To find Eigenvectors

Step 4:- If the Eigenvectors are orthogonal then form a normalized model matrix N

Step 5:- Find N^T

Step 6:- Calculate AN

Step 7:- Calculate $D = N^T AN$

problems:

① Symmetric matrix [$N^T AN$] Diagonalise the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Soln:-

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Step:-1

To find the char. eqn

The char. equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 8 + 7 + 3$$

$$= 18$$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 5 + 20 + 20$$

$$= 45$$

$$S_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) + 6(-18+8) + 2(24-14)$

$= 8(5) + 6(-10) + 2(10)$

$= 40 - 60 + 20$

$= 60 - 60$

$= 0$

$\therefore$ The char. eqn is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

Step:- 2

To solve the characteristic.

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\lambda = 0, \ \lambda = 3, \ \lambda = 15$$

Hence the Eigenvalues are $\lambda_1 = 0, \ \lambda_2 = 3, \ \lambda_3 = 15$

Step:- 3

To find the Eigenvectors

To find the Eigenvectors solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— (A)}$$

Case (i)

when $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$8x_1 - 6x_2 + 2x_3 = 0$ ——— (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ ——— (2)

$2x_1 - 4x_2 + 3x_3 = 0$ ——— (3)

Solving (1) & (2) we get

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{} = \frac{x_2}{} = \frac{x_3}{}$$

Hence the Corresponding Eigenvector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii)

When $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$5x_1 - 6x_2 + 2x_3 = 0$ —— (4)

$-6x_1 + 4x_2 - 4x_3 = 0$ —— (5)

$2x_1 - 4x_2 + 0x_3 = 0$ —— (6)

Solving (5) & (6) we get

$$\frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the Corresponding Eigenvector $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case (iii)

When $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-7x_1 - 6x_2 + 2x_3 = 0$ —— (7)

$-6x_1 - 8x_2 - 4x_3 = 0$ —— (8)

$2x_1 - 4x_2 - 12x_3 = 0$ —— (9)

Solving (8) and (9)

$$\frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16}$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence the Corresponding Eigenvector $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ The set of Eigenvectors are

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$x_1^T x_2 = 2+2-4 = 0$$

$$x_1^T x_3 = 2-4+2 = 0$$

$$x_2^T x_3 = 4-2-2 = 0$$

Hence the Eigenvectors are orthogonal to each other.

Step 4:-

To form the normalised matrix N.

$$N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = 1/3 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Step:-5

Find N^T

$$N^T = 1/3 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Step 6:-

Calculate AN

$$AN = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} 1/3 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

Step:-7 Calculate $N^T AN$

$$N^T AN = 1/3 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 0 & 2+2-4 & 10-20+10 \\ 0 & 4+1+4 & 20-10-10 \\ 0 & 4-2-2 & 20+20+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$\therefore D = N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Home work:-

$\lambda = 1, -1, 4$

① $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ Ans:- $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

③ $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Ans

② $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ Ans:- $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

QUADARATIC FORM - ORTHOGONAL REDUCTION TO ITS CANONICAL FORM

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2}\text{coeff } x_1x_2 & \frac{1}{2}\text{coeff } x_1x_3 \\ \frac{1}{2}\text{coeff } x_1x_2 & \text{coeff } x_2^2 & \frac{1}{2}\text{coeff } x_2x_3 \\ \frac{1}{2}\text{coeff } x_1x_3 & \frac{1}{2}\text{coeff } x_2x_3 & \text{coeff } x_3^2 \end{bmatrix}$$

① Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

Soln

Here $x_2x_1 = x_1x_2$

$x_3x_1 = x_1x_3$

$x_2x_3 = x_3x_2$

$$\therefore Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

① Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy - 10xz - 2yz$

$$Q = \begin{bmatrix} 2 & 2 & -5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

Nature of a quadratic form.

let $D_1 = |a_{11}| = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$D_1\ D_2 \ldots D_n$ are the principal minors of A.

(i) The Q.F is positive definite if $D_1\ D_2 \ldots D_n$ are all +ve

(or) $D_n > 0$ for all n.

(ii) The Q.F is negative definite if $D_1, D_3, D_5 \ldots$ are all negative and $D_2, D_4, D_6 \ldots$ are all positive.

(or) $(-1)^n D_n > 0$ for all n.

(iii) The Q.F is positive semi-definite if $D_n \geq 0$ and atleast one $D_i = 0$

(iv) The Q.F is negative semi-definite if $(-1)^n D_n \geq 0$ and atleast one $D_i = 0$

(v) The Q.F is indefinite in all other cases.

Problems:-

① Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$

Soln: The matrix of the Q.F is

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$D_1 = |1| = 1$ (+ve)

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \ (+ve)$$

$$|B_3| = 0$$

$\therefore$ The Q.F is said to be positive semi-definite.

② prove that the Q.f $x^2+2y^2+3z^2+2xy+2yz-2xz$ is indefinite.

Soln

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$D_1 = |1| = 1 \ (+ve)$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \ (+ve)$$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 1(6-1) - 1(3+1) - 1(1+2)$$

$$= 1(5) - 1(4) - 1(3) = 5-4-3 = -2 \ (-ve)$$

$\therefore$ The Q.F is indefinite.

③ Discuss the nature of the Q.F $2x^2+3y^2+2z^2+2xy$

$$Q = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D_1 = |2| = 2 \ (+ve)$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (6-1) = 5 \ (+ve)$$

$$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0(0-0)$$

$$= 2(6) - 1(2) = 12-2$$

$$= 10 \ (+ve)$$

$\therefore$ The Q.F is Positive definite.

<u>Reduction of Quadratic form to canonical form through orthogonal transformation</u>

<u>working rule.</u>

Step 1:- write the matrix of the given Q.F

Step 2:- To find the char. equation

Step 3:- To solve the char. equation

Step 4:- To find the Eigenvectors orthogonal to each other

Step 5:- Form Normalised matrix N.

Step 6:- Find N^T

Step 7:- Find AN

Step 8:- Find $D = N^T AN$

Step 9:- Canonical form $[y_1 \; y_2 \; y_3] \, [D] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Problems:-

1) Reduce the Quadratic form $Q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$ into Canonical form by an orthogonal transformation.

Soln:- Given $6x^2+3y^2+3z^2-4xy-2yz+4zx$

Step 1:- The matrix of the Q.F is $A = \begin{bmatrix} \text{coeff } x^2 & \frac{1}{2}\text{ coeff } xy & \frac{1}{2}\text{ coeff } xz \\ \frac{1}{2}\text{coeff } yx & \text{Coeff } y^2 & \frac{1}{2}\text{ Coeff } yz \\ \frac{1}{2}\text{ Coeff } zx & \frac{1}{2}\text{Coeff } zy & \text{Coeff } z^2 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Step:2

To find the char. eqn of A

The chala. eqn of A is $|A-\lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = 6+3+3 = 12$

$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= (9-1)+(18-4)+(18-4)$

$= 8+14+14 = 36$

$$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32$$

$\therefore$ The char. eqn is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

Step 3:-

To solve the char. equation

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

If $\lambda = 1$ then $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 1 - 12 + 36 - 32 \neq 0$

If $\lambda = -1$ then $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = -1 - 12 - 36 - 32 \neq 0$

If $\lambda = 2$ then $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 8 - 48 + 72 - 32 = 0$

$\therefore \lambda = 2$ is a root.

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0$$

$$\lambda = 8, \lambda = 2$$

Eigenvalues are 2, 2, 8

Step 4

To get the Eigenvectors solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{—— (A)}$$

Scanned by CamScanner

Case (i) If $\lambda = 8$ then the eqn (A) becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{——} (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{——} (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \text{——} (3)$$

Solving (1) & (2) we get

$$\frac{x_1}{2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii)

When $\lambda = 2$ then the eqn (A) becomes

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \text{——} (4)$$

$$-2x_1 + x_2 - x_3 = 0 \quad \text{——} (5)$$

$$2x_1 - x_2 + x_3 = 0 \quad \text{——} (6)$$

(4) (5) (6) are same as

$$2x_1 - x_2 + x_3 = 0$$

If $x_1 = 0$ we get $-x_2 + x_3 = 0$

$$-x_2 = -x_3$$

$$x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

To find the third Eigenvector orthogonal to x_1 and x_2 since the matrix A is symmetric

$$\therefore 2l - m + n = 0$$
$$0.l + m + n = 0$$
$$\frac{l}{-1-1} = \frac{-m}{2-0} = \frac{n}{2-0}$$
$$\frac{l}{-2} = \frac{m}{-2} = \frac{n}{2}$$
$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$$

Hence the Eigenvector $x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Eigenvector	normalized form
$x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
$x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$
$x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

Step:- 5

Form Normalised matrix N

$$N = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

Step:- 6

Find N^T

$$N^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -1/\sqrt{3} \end{bmatrix}$$

Step:- 7

Find AN

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12+2+2}{\sqrt{6}} & \frac{0-2+2}{\sqrt{2}} & \frac{6-2-2}{\sqrt{3}} \\ \frac{-4-3-1}{\sqrt{6}} & \frac{0+3-1}{\sqrt{2}} & \frac{-2+3+1}{\sqrt{3}} \\ \frac{4+1+3}{\sqrt{6}} & \frac{0-1+3}{\sqrt{2}} & \frac{2-1-3}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ -\frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{3}} \end{bmatrix}$$

Step:- 8 :-

Find N^TAN

$$D = N^TAN = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ -\frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & \frac{0-2+2}{\sqrt{12}} & \frac{4-2-2}{\sqrt{18}} \\ \frac{0-8+8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0+2-2}{\sqrt{6}} \\ \frac{16-8-8}{\sqrt{18}} & \frac{0+2-2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 9:-

Canonical form

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow 8y_1^2 + 2y_2^2 + 2y_3^2$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Home work:-

① Reduce the quadratic form to a canonical form by an orthogonal reduction $2x_1x_2 + 2x_1x_3 + 2x_2x_3$ also discuss its nature.

② Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to the canonical form through an orthogonal transformation and hence show that it is positive semi-definite.

UNIT-II VECTOR CALCULUS

GREEN'S THEOREM IN A PLANE

If u, v $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ are continuous and one-valued functions in the region R enclosed by the curve C, then

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Problems:-

① Verify Green's theorem in the plane for $\int_C (x^2 dx + xy\,dy)$ where C is the curve in the xy plane given by $x=0$, $y=0$, $x=a$, $y=a$ $(a>0)$

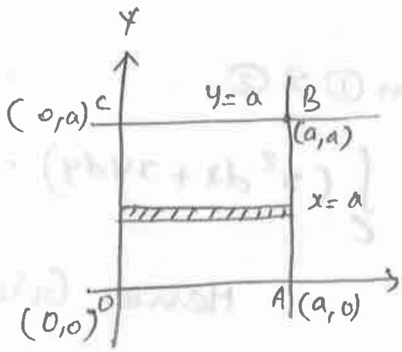
Soln:-

Here $u = x^2$ $\qquad v = xy$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = y$$

To evaluate $\int_C (x^2 dx + xy\,dy) = \int_{OA}(x^2 dx + xy\,dy) + \int_{AB}(x^2 dx + xy\,dy) + \int_{BC}(x^2 dx + xy\,dy)$

$$+ \int_{CO}(x^2 dx + xy\,dy)$$



Along OA

$y = 0$

$dy = 0$

x varies from $x=0$ to $x=a$

$$\int_{OA}(x^2 dx + xy\,dy) = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB

$x = a$

$dx = 0$

y varies from $y=0$ to $y=a$

$$\therefore \int_{AB}(x^2 dx + xy\,dy) = \int_0^a ay\,dy = \left[a\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

Along BC

$y = a$

$dy = 0$

x varies from $x=a$ to $x=0$

$$\int_{BC}(x^2 dx + xy\,dy) = \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = \left[0 - \frac{a^3}{3}\right] = -\frac{a^3}{3}$$

Along CO

$x=0$

$dx=0$

y varies from $y=a$ to $y=0$

$$\therefore \int_{CO}(x^2dx+xydy) = \int_a^0 0\,y\,dy = 0$$

$$\therefore \int_C (x^2dx+xydy) = a^3/3 + a^3/2 - a^3/3 + 0 = a^3/2 \quad \text{—— (1)}$$

R.H.S

To evaluate $\iint_R \left(\frac{\partial V}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$

$$\iint_R (y-0)\,dx\,dy = \iint_R y\,dx\,dy = \int_0^a\int_0^a y\,dx\,dy$$

$$= \int_0^a [yx]_0^a\,dy = \int_0^a ay\,dy$$

$$= \left[a\,y^2/2\right]_0^a = a^3/2 \quad \text{—— (2)}$$

from (1) & (2)

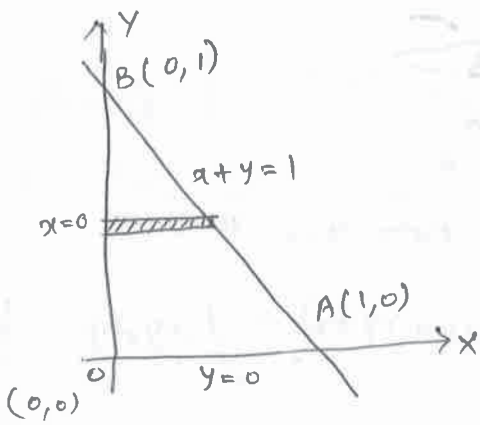
$$\int_C (x^2dx+xydy) = \iint_R \left(\frac{\partial V}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

Hence Green's theorem is verified.

(2) Verify Green's theorem in the xy plane for $\int_C \{(3x-8y^2)dx + (4y-6xy)dy\}$ where c is the boundary of the region given by $x=0,\ y=0,\ x+y=1$

Soln:-

Green's theorem is



$$\int_C (udx+Vdy) = \iint_R \left(\frac{\partial V}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

Here $u = 3x-8y^2$ $\qquad V = 4y-6xy$

$\frac{\partial u}{\partial y} = -16y$ $\qquad \frac{\partial V}{\partial x} = -6y$

To evaluate $\int_C (udx+Vdy) = \int_{OA}(udx+Vdy) + \int_{AB}(udx+Vdy) + \int_{BO}(udx+Vdy)$

(i) Along OA

$y=0,\ dy=0$

$\int_{OA} \{(3x-8y^2)\,dx + (4y-6xy)\,dy\}$; x varies from 0 to 1

$= \int_{OA} 3x\,dx$

$= 3\int_0^1 x\,dx = 3\left[\frac{x^2}{2}\right]_0^1 = 3\left[\frac{1}{2}\right] = \frac{3}{2}$

(ii) Along AB $(x+y=1)$

$\int_{AB} (3x-8y^2)\,dx + (4y-6xy)\,dy$

$x+y=1$

$y=1-x$

$dy=-dx.$

x varies from 1 to 0

$= \int_{AB} (3x-8(1-x)^2)\,dx + \{4(1-x)-6x(1-x)\}(-dx)$

$= \int_{AB} (3x-8(1+x^2-2x))\,dx + (4-4x-6x+6x^2)(-dx)$

$= \int_1^0 (3x-8-8x^2+16x-4+4x+6x-6x^2)\,dx \rightarrow \int_1^0 [-14x^2+29x-12]\,dx$

$= \left[-14\frac{x^3}{3}+29\frac{x^2}{2}-12x\right]_1^0 = \left[(-0+0-0)-\left(\frac{-14}{3}+\frac{29}{2}-12\right)\right]$

$= \frac{14}{3}-\frac{29}{2}+12 = \frac{13}{6}$

(iii) Along BO $(x=0)$ $[x=0, dx=0]$

$\int_{BO} (3x-8y^2)\,dx + (4y-6xy)\,dy$

$= \int_1^0 4y\,dy = 4\int_1^0 y\,dy = 4\left[\frac{y^2}{2}\right]_1^0 = 0-\frac{4}{2} = -2.$

Here $\int_C (u\,dx+v\,dy) = \int_{OA}+\int_{AB}+\int_{BO} = \frac{3}{2}+\frac{13}{6}-2 = \frac{10}{6} = \frac{5}{3}$ —— (1)

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy$

$\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy = \iint_R (-6y+16y)\,dx\,dy$

$= \iint_R 10y\,dx\,dy = 10\int_0^1\int_0^{1-y} y\,dx\,dy$

$= 10\int_0^1 [yx]_0^{1-y}\,dy$

$= 10\int_0^1 [y(1-y)]\,dy = 10\int_0^1 (y-y^2)\,dy$

$$= 10\int_0^1 (y-y^2)\,dy = 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 10\left[\frac{1}{2}-\frac{1}{3}\right]$$

$$= 10\left[\frac{1}{6}\right] = \frac{5}{3} \quad \text{—— (2)}$$

From (1) & (2)

L.H.S = R.H.S

Hence Green's theorem is verified.

(3) Verify Green's theorem in the xy plane for $\int_C \{(xy+y^2)\,dx + x^2\,dy\}$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

Soln:-

Green's theorem in the xy plane is

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Here $u = xy + y^2$ $\qquad v = x^2$

$$\frac{\partial u}{\partial y} = x + 2y \qquad \frac{\partial v}{\partial x} = 2x$$

Evaluation of $\int_C (u\,dx + v\,dy)$

$$\int_C (u\,dx + v\,dy) = \int_{OA} (u\,dx + v\,dy) + \int_{AO} (u\,dx + v\,dy)$$

(i) Along OA $[y = x^2]$

$y = x^2$

$dy = 2x\,dx$

x varies from 0 to 1

$$\int_0^1 (x^3 + x^4)\,dx + 2x^3\,dx = \int_0^1 (3x^3 + x^4)\,dx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \left[\frac{3}{4} + \frac{1}{5}\right] = \frac{19}{20}$$

(ii) Along AO $[y = x]$

$y = x$

$dy = dx$

$$\int_{AO} = \int_{AO} (x^2 + x^2)\,dx + x^2\,dx.$$

x varies from 1 to 0

$$= \int_1^0 3x^2\,dx = 3\left[\frac{x^3}{3}\right]_1^0 = \left[x^3\right]_1^0 = 0 - 1 = -1$$

Hence $\int_C \{(xy+y^2)\,dx + x^2\,dy\} = \int_{OA} + \int_{AO} = \frac{19}{20} - 1 = -\frac{1}{20}$ —— (1)

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy = \iint_R [2x-(x+2y)]\,dx\,dy$$

$$= \int_0^1\int_y^{\sqrt{y}} (x-2y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy = \int_0^1 \left[\left(\frac{y}{2} - 2y^{3/2}\right) - \left(\frac{y^2}{2} - 2y^2\right)\right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} + \frac{3}{2}y^2\right]dy$$

$$= \left[\frac{y^2}{4} - 2\,\frac{y^{5/2}}{5/2} + \frac{3}{2}\,\frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1}{4} - \frac{4}{5} + \frac{1}{2}\right] - (0-0+0)$$

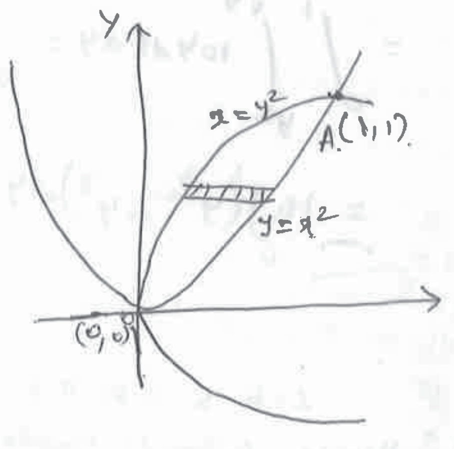
$$= \frac{5-16+10}{20} = \frac{15-16}{20} = \frac{-1}{20} \quad \text{--- (2)}$$

∴ L.H.S = R.H.S

Hence Green's theorem is verified.

(4) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx+(4y-6xy)\,dy$ where C is the boundary of the region defined by $x=y^2$, $y=x^2$

Soln:



Here $u = 3x^2-8y^2$ $\quad v = 4y-6xy$

$\frac{\partial u}{\partial y} = -16y$ $\quad \frac{\partial v}{\partial x} = -6y$

Evaluation of $\int_C u\,dx+v\,dy$

$$\int_C u\,dx+v\,dy = \int_{OA} u\,dx+v\,dy + \int_{AO} u\,dx+v\,dy$$

(i) Along OA. [$y=x^2$]

$y = x^2$

$dy = 2x\,dx$.

x varies from 0 to 1

$$\int_0^1 (3x^2-8x^4)\,dx + (4x^2-6x^3)\,2x\,dx$$

$$\int_0^1 [3x^2-8x^4+8x^3-12x^4]\,dx.$$

$$= \int_0^1 [-20x^4 + 8x^3 + 3x^2]\, dx$$

$$= \left[-\cancel{20}^{4}\frac{x^5}{\cancel{5}} + \cancel{8}^{2}\frac{x^4}{\cancel{4}} + \cancel{3}\frac{x^3}{\cancel{3}}\right]_0^1$$

$$= \left[-4x^5 + 2x^4 + x^3\right]_0^1 = [-4 + 2 + 1] = -4 + 3 = -1.$$

(ii) Along AO $[\because y^2 = x \quad 2y\,dy = dx]$

y varies from 1 to 0

$$\int_1^0 (3y^4 - 8y^2)\, 2y\, dy + (4y - 6y^3)\, dy$$

$$\int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\, dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\, dy$$

$$= \left(\cancel{6}\frac{y^6}{\cancel{6}} - \cancel{22}\frac{y^4}{\cancel{4}} + \cancel{4}\frac{y^2}{\cancel{2}}\right)_1^0$$

$$= \left[y^6 - \tfrac{11}{2}y^4 + 2y^2\right]_1^0 = (0 - 0 + 0) - (1 - \tfrac{11}{2} + 2) = -(3 - \tfrac{11}{2}) = -(-\tfrac{5}{2}) = \tfrac{5}{2}$$

$$\int_C u\,dx + v\,dy = \int_{OA} + \int_{AO} = -1 + \tfrac{5}{2} = \tfrac{3}{2} \quad —①$$

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\, dx\, dy$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\, dx\, dy = \int_0^1 [10xy]_{y^2}^{\sqrt{y}}\, dy = \int_0^1 10y(\sqrt{y} - y^2)\, dy$$

$$= 10\int_0^1 (y^{3/2} - y^3)\, dy = 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left[\left(\tfrac{2}{5} - \tfrac{1}{4}\right) - (0 - 0)\right] = 10\left[\frac{8-5}{20}\right] = \tfrac{3}{2} \quad —②$$

L.H.S = R.H.S

Hence Green's theorem is verified.

⑤ verify Green's theorem in a plane for the integral $\int_C (x - 2y)\, dx + x\, dy$ taken round the circle $C \equiv x^2 + y^2 = 1$

Soln::

$u = x - 2y$ and $v = x$. and C is the circle $x^2 + y^2 = 1$

we know that parametric equation of the circle

$x = \cos\theta \qquad y = \sin\theta.$

$dx = -\sin\theta\, d\theta \qquad dy = \cos\theta\, d\theta$

clearly θ varies from 0 to 2π

To find $\int_C (u\,dx + v\,dy)$

$$\int_C u\,dx+v\,dy = \int_0^{2\pi}\left[(\cos\theta - 2\sin\theta)(-\sin\theta\,d\theta) + (\cos\theta)\cos\theta\,d\theta\right]$$

$$= \int_0^{2\pi}\left[\cos\theta\sin\theta + 2\sin^2\theta + \cos^2\theta\right]d\theta$$

$$= \int_0^{2\pi}\left[-\frac{\sin 2\theta}{2} + 2\left(\frac{1-\cos 2\theta}{2}\right) + \left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[-\sin 2\theta + 2 - 2\cos 2\theta + 1+\cos 2\theta\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[3 - \sin 2\theta - \cos 2\theta\right]d\theta$$

$$= \frac{1}{2}\left[3\theta + \frac{\cos 2\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[3(2\pi) + \frac{\cos 2(2\pi)}{2} - \frac{\sin 2(2\pi)}{2}\right] - \left[0 + \frac{\cos 0}{2} - \frac{\sin 0}{2}\right]$$

$$= \frac{1}{2}\left[6\pi + \frac{1}{2} - 0\right] - \left[0 + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[6\pi + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[6\pi\right] = 3\pi$$

To find $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$

$$\iint_R \left[\frac{\partial(x)}{\partial x} - \frac{\partial(x-2y)}{\partial y}\right]dx\,dy$$

$$= \iint_R (1+2)\,dx\,dy$$

$$= 3\iint_R dx\,dy = 3\,[\text{Area of the circle}]$$

$$= 3\pi r^2$$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy = 3\pi \quad (\because \text{radius} = 1) \quad \text{— ②}$$

From ① & ② we see that

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

STOKE'S THEOREM STATEMENT - VERIFICATION AND ITS APPLICATIONS.

Statement:-

The surface integral of the normal component of the curl of a vector function F over an open surface S is equal to the line integral of the tangential component of F around the closed curve C bounding S

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

Problems:-

① Verify Stroke's theorem for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the XOY plane bounded by the lines $x=0, x=a$ $y=0$ and $y=b$.

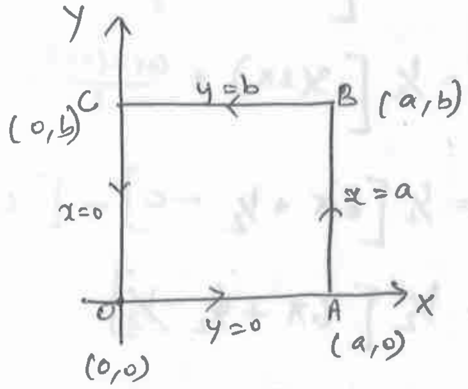
Soln:-

Stroke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$

$\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j} + 0\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}\cdot d\vec{r} = (x^2-y^2)dx + 2xy\, dy$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$= i(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y)$

$= 4y\vec{k}$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$

(i.e.) $\hat{n} = \vec{k}$

$$\therefore R.H.S = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \iint_S 4y\vec{k}\cdot\vec{k}\, ds$$

$$= \iint_S 4y\, dx dy = \int_0^b\int_0^a 4y\, dx\, dy$$

$$= 4\int_0^b (xy)_0^a dy = 4\int_0^b ay\, dy = 4\left[a\frac{y^2}{2}\right]_0^b = 4\left[\frac{ab^2}{2}\right] = 2ab^2$$

$$L.H.S = \int_C \vec{F}\cdot d\vec{r} = \int_C (x^2-y^2)dx + 2xy\, dy$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\int_{OA} = \int_{OA} (x^2-y^2)\,dx + 2xy\,dy \qquad [y=0,\ dy=0]$$

$$= \int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\int_{AB} = \int_{AB} (x^2-y^2)\,dx + 2xy\,dy \qquad [x=a\ \ dx=0]$$

$$= \int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = \cancel{2}a\left[\frac{b^2}{\cancel{2}}\right] = ab^2$$

$$\int_{BC} = \int_{BC} (x^2-y^2)\,dx + 2xy\,dy \qquad [y=b,\ dy=0]$$

$$= \int_a^0 (x^2-b^2)\,dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0 = -\frac{a^3}{3} + ab^2$$

$$\int_{CO} = \int_{CO} (x^2-y^2)\,dx + 2xy\,dy \qquad [x=0,\ dx=0]$$

$$= \int_{CO} 0$$

$$\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2$$

② verify Stroke's theorem for $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$ where s is the surface bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ $z=0$ and $z=1$ above the xoy plane.

Soln: Stroke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$

$$\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$$

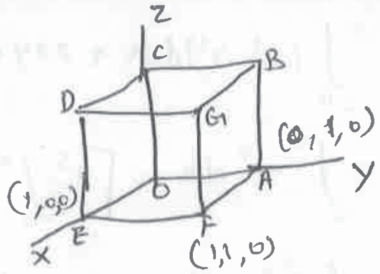
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (y-z)\,dx + yz\,dy - xz\,dz$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z & yz & -xz \end{vmatrix}$$

$$= \vec{i}\,(0-y) - \vec{j}\,(-z-(-1)) + \vec{k}\,(0-1)$$

$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$



$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

$\iint_{S_6}$ is not applicable.

Since the given condition is above the xoy plane.

$$\iint_{S_1} = \iint_{OCDE} \left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\cdot\vec{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 -y\,dy\,dz$$

$$= \int_0^1 \left[-\frac{y^2}{2}\right]_0^1 dz = \int_0^1 -\frac{1}{2}dz = \left[-\frac{1}{2}z\right]_0^1 = -\frac{1}{2}$$

$$\iint_{S_2} = \iint_{ABGF} \left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\cdot(-\vec{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 y\,dy\,dz = \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dz = \int_0^1 \frac{1}{2}dz = \frac{1}{2}[z]_0^1 = \frac{1}{2}[1-0] = \frac{1}{2}$$

$$\iint_{S_3} = \iint_{ABCO} \left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\cdot(\vec{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 (z-1)\,dx\,dz = \int_0^1 [zx-x]_0^1 dz = \int_0^1 [z-1]\,dz = \left[\frac{z^2}{2}-z\right]_0^1 = \frac{1}{2}-1 = -\frac{1}{2}$$

$$\iint_{S_4} = \iint_{DGEF} \left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\cdot(-\vec{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 [-(z-1)]\,dx\,dz$$

$$= -\int_0^1 (zx-x)_0^1\,dz = -\int_0^1 (z-1)\,dz = -\left[\frac{z^2}{2}-z\right]_0^1 = -\left[\frac{1}{2}-1\right] = -\left[-\frac{1}{2}\right] = \frac{1}{2}$$

$$\iint_{S_5} = \iint_{CDBG} \left[-y\vec{i}+(z-1)\vec{j}-\vec{k}\right]\cdot(\vec{k})\,dx\,dy$$

$$= \int_0^1\int_0^1 (-1)\cdot dx\,dy$$

$$= \int_0^1 (-x)_0^1 dy = \int_0^1 [1-0]\,dy = \int_0^1 -dy = -[1-0] = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} = -\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-1 = -1$$

R.H.S $= -1$.

L.H.S $\int_C \vec{F}\cdot d\vec{r} = \int_{OE} + \int_{EF} + \int_{FA} + \int_{AO}$

$$\int_{OE} = \int_{OE} (y-z)\,dx + yz\,dy - xz\,dz$$

$$= \int_{OE} 0 \qquad [y=0,\ z=0,\ dy=0\ dz=0]$$

$$= 0$$

$$\int_{EF} = \int_{EF} (y-z)\,dx + yz\,dy - xz\,dz$$

$[x=1,\ z=0,\ dx=0\ dz=0]$

$$= \int_{EF} 0$$

$$\int_{FA} = \int_{FA} (y-z)\,dx + yz\,dy - xz\,dz \qquad [y=1,\ z=0\ \ dy=0\ \ dz=0]$$

$$= \int_1^0 1\,dx$$

$$= [x]_1^0 = 0-1 = -1$$

$[x=0,\ z=0,\ dx=0,\ dz=0]$

$$\int_{AO} = \int_{AO} (y-z)\,dx + yz\,dy - xz\,dz$$

$$= \int_{AO} 0$$

$$= 0$$

$$\therefore \int_C = \int_{OE} + \int_{EF} + \int_{FA} + \int_{AO}$$

$$= 0+0-1+0 = -1$$

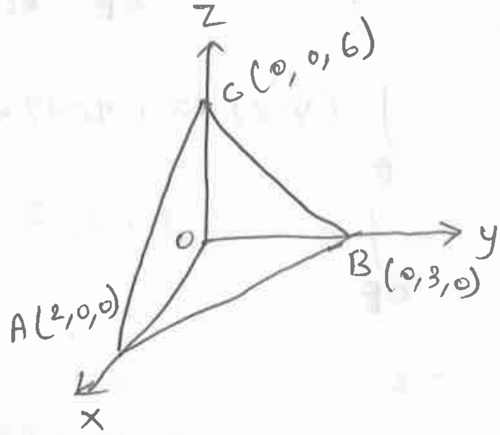
L.H.S $= -1$

L.H.S = R.H.S

③ Evaluate the integral $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$ where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ using Stoke's theorem

Soln: $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$ where S is the surface of the triangle and $\hat{n}$ is the unit vector normal to surface S.

$$\vec{F}\cdot d\vec{r} = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$



$$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$$

and $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= i\,(1+1) - j\,(0-0) + \vec{k}\,(2-1)$$

$$\nabla\times\vec{F} = 2\vec{i} + \vec{k}$$

Equation of the plane ABC is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$$3x + 2y + z = 6$$

Let $\phi = 3x + 2y + z - 6$

$$\nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$$

unit normal vector to the surface ABC (or ϕ) is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}$$

$$\nabla\times\vec{F}\cdot\hat{n} = (2\vec{i}+\vec{k})\cdot\left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right) = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

Here $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_S \frac{7}{\sqrt{14}}\,ds = \frac{7}{\sqrt{14}}\iint_R \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$

where R is the projection of surface ABC on xoy plane

$$= \frac{7}{\sqrt{14}}\iint_R \frac{dx\,dy}{\frac{1}{\sqrt{14}}} \qquad \left[\hat{n}\cdot\vec{k} = \left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right)\vec{k} = \frac{1}{\sqrt{14}}\right]$$

$$= 7\iint_R dx\,dy$$

$$= 7\times(\text{Area of triangle ABC}) \quad \tfrac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$$

$$= 7\times 3$$

$$= 21$$

Note:- Let R be the projection of the surface on the XY plane, YZ plane and XZ plane

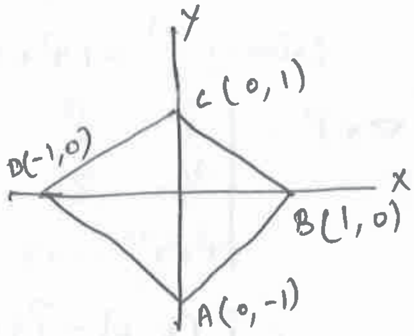
① $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$ ② $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\,dy\,dz$, ③ $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|}\,dz\,dx$.

(4) Evaluate $\int_C (xy\,dx + xy^2\,dy)$ by Stoke's theorem where C is the square in xy plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ and $(0,-1)$. (13)

Soln:

Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \quad \text{——} (1)$$



Given $\vec{F}\cdot d\vec{r} = xy\,dx + xy^2\,dy$

$\therefore \vec{F} = xy\,\vec{i} + xy^2\,\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$$\text{curl}\,\vec{F} \text{ (or) } \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y^2 - x)$$

$$= (y^2 - x)\vec{k}$$

Here ABCD is a square which lies on xoy plane. Hence the unit normal vector to this surface (square) is $\vec{k}$

$$\hat{n}\,ds = \vec{k}\,dx\,dy$$

$$\nabla\times\vec{F}\cdot\hat{n}\,ds = (y^2 - x)\vec{k}\cdot\vec{k}\,dx\,dy$$

$$= (y^2 - x)\,dx\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (y^2 - x)\,dx\,dy$$

where R is the region in the xoy plane which is bounded by the square ABCD.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\,dx\,dy$$

$$= \int_{-1}^{1} \left[y^2 x - \frac{x^2}{2}\right]_{-1}^{1} dy$$

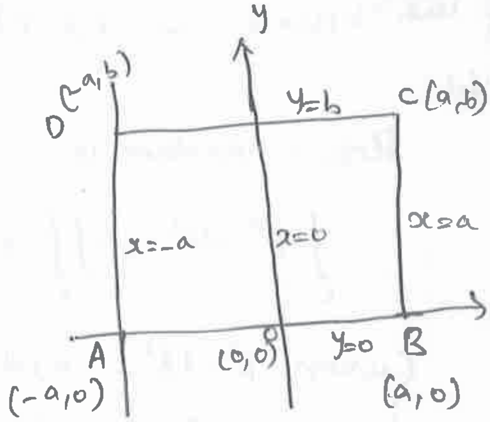
$$= \int_{-1}^{1} \left[\left(y^2 - \tfrac{1}{2}\right) - \left(-y^2 - \tfrac{1}{2}\right)\right] dy$$

$$= \int_{-1}^{1} \left[y^2 - \tfrac{1}{2} + y^2 + \tfrac{1}{2}\right] dy = \int_{-1}^{1} 2y^2\,dy = 2\left[\frac{y^3}{3}\right]_{-1}^{1} = 2\left[\tfrac{1}{3} + \tfrac{1}{3}\right]$$

$$= 4/3 \text{ //Ans}$$

⑤ Verify Stokes theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$

Soln:



Stroke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$

Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y]$$

$$= 0\vec{i} - 0\vec{j} - 4y\vec{k}$$

$$= -4y\vec{k}$$

$$R.H.S = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_S (-4y\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \iint_S -4y\,dx\,dy = \int_0^b\int_{-a}^{a} -4y\,dx\,dy$$

$$= -4\int_0^b [yx]_{-a}^{a}\,dy$$

$$= -4\int_0^b [ay+ay]\,dy = -4\int_0^b 2ay\,dy$$

$$= -8a\int_0^b y\,dy = -8a\int_0^b y\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4a[b^2]$$

$$= -4ab^2$$

Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)dx - 2xy\,dy$$

$$L.H.S = \int_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$\int_{AB} (x^2+y^2)\,dx - 2xy\,dy = \int_{-a}^{a} (x^2+y^2)dx - 2xy\,dy \quad [y=0 \quad dy=0]$$

$$= \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \left[\frac{a^3}{3} + \frac{a^3}{3}\right] = \frac{2a^3}{3}$$

$$\int_{BC} (x^2+y^2)\,dx - 2xy\,dy = \int_0^b (x^2+y^2)\,dx - 2xy\,dy \quad [x=a \quad dx=0]$$

$$= \int_0^b -2ay\,dy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -\cancel{2}a\left[\frac{b^2}{\cancel{2}}\right] = -ab^2$$

$$\int_{CD} (x^2+y^2)\,dx - 2xy\,dy = \int_a^{-a} (x^2+y^2)\,dx - 2xy\,dy \quad [y=b \quad dy=0]$$

$$= \int_a^{-a} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

$$\int_{DA} (x^2+y^2)\,dx - 2xy\,dy = \int_b^0 2ay\,dy \quad [x=-a \quad dx=0]$$

$$= 2a\int_b^0 y\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = 2a\left[0 - \frac{b^2}{2}\right] = \cancel{2}a\left[\frac{-b^2}{\cancel{2}}\right] = -ab^2$$

$$\therefore \int_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= \cancel{\frac{2a^3}{3}} - ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 = -4ab^2$$

L.H.S = R.H.S Hence stoke's theorem verified.

(2) Using stoke's theorem evaluate $\int_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$ where C is the circle $x^2+y^2=1$ corresponding to the surface of the sphere of unit radius.

Soln:
we know that stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Given $\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz$

$$\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\vec{i} - \vec{j}(0-0) + (0+1)\vec{k}$$

$\nabla \times \vec{F} = \vec{k}$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$$= \iint_S \vec{k} \cdot \hat{n}\, ds$$

$$= \iint_R \vec{k} \cdot \hat{n} \frac{dx\,dy}{|\vec{k} \cdot \hat{n}|}$$ where R is the Projection of S in the xy plane.

$$= \iint_R dx\,dy$$

= area of unit circle

$= \pi$

GAUSS DIVERGENCE THEOREM.

Statement:-
The Surface integral of the normal component of a vector function F over a closed surface S enclosing volume V is equal to the volume integral of the divergence of F taken throughout the volume V.

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

Problems:-

① Verify the Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

Soln:-
Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla \cdot \vec{F} = 4z - 2y + y = 4z - y$$

Now $$\iiint_V \nabla \cdot \vec{F}\, dV = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz$$

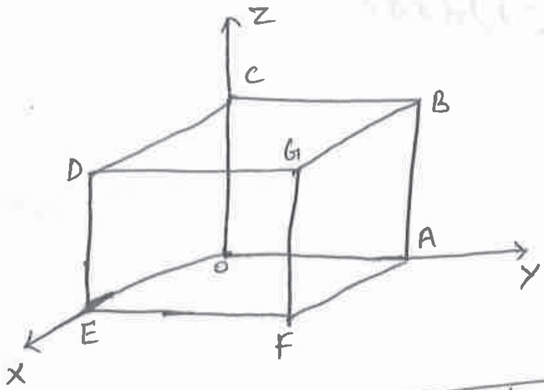
$$= \int_0^1\int_0^1 [4zx - yx]_0^1\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z - y)\, dy\, dz$$

$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right) dz = \left(\frac{4z^2}{2} - \frac{1}{2}z\right)_0^1$$

$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Now $\iint\limits_S \vec{F}\cdot\hat{n}\,ds = \iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} + \iint\limits_{S_4} + \iint\limits_{S_5} + \iint\limits_{S_6}$



		unit outward normal vector ($\hat{n}$)	
S_1	OCDE	$\vec{i}$	$x=1$
S_2	ABGF	$-\vec{i}$	$x=0$
S_3	OCBA	$\vec{j}$	$y=1$
S_4	GDEF	$-\vec{j}$	$y=0$
S_5	CDGB	$\vec{k}$	$z=1$
S_6	OEFA	$-\vec{k}$	$z=0$

(i) $\iint\limits_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{OCDE} \vec{F}\cdot\vec{i}\,dy\,dz$

$= \int_0^1\int_0^1 (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{i}\,dy\,dz$

$= \int_0^1\int_0^1 4xz\,dy\,dz \qquad [x=1 \text{ on } S_1]$

$= \int_0^1\int_0^1 4z\,dy\,dz = 4\int_0^1 [zy]_0^1\,dz = 4\int_0^1 z\,dz = 4\left[\frac{z^2}{2}\right]_0^1 = 4[\tfrac{1}{2}-0]$
$= 2$

(ii) $\iint\limits_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{ABGF} \vec{F}\cdot(-\vec{i})\,dy\,dz$

$= \int_0^1\int_0^1 (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i})\,dy\,dz \qquad [x=0 \text{ on } S_2]$

$= \int_0^1\int_0^1 -4xz\,dy\,dz$

$= 0$

(iii) $\iint\limits_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{OCBA} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}\,(dx\,dz)$

$[y=1 \text{ on } S_3]$

$= \int_0^1\int_0^1 -y^2\,dx\,dz$

$= \int_0^1\int_0^1 -dx\,dz = -\int_0^1 [x]_0^1\,dz = -\int_0^1 dz = -[1-0] = -1$

(iv) $\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{GDEF} \vec{F}\cdot(-\vec{j})\,dx\,dz$ [$y=0$ on S_4]

$$= \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot(-\vec{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz$$

$$= 0$$

(v) $\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{CDGB} \vec{F}\cdot(\vec{k})\,dx\,dy$

$$= \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot\vec{k}\,dx\,dy$$

[$z=1$ on S_5]

$$= \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 [yx]_0^1\,dy$$

$$= \int_0^1 [y]\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

(vi) $\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFA} \vec{F}\cdot(-\vec{k})\,dx\,dy$

$$= \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot(-\vec{k})\,dx\,dy$$

[$z=0$ on S_6]

$$= \int_0^1\int_0^1 -yz\,dx\,dy$$

$$= 0$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0$$

$$= 2 - 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$\therefore$ L.H.S = R.H.S

② Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

Soln:

Gauss divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z$$

$$R.H.S = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$= \int_0^a \int_0^b \int_0^c 2(x+y+z)\, dz\, dy\, dx$$

$$= 2\int_0^a \int_0^b \left(xz + yz + z^2/2\right)_0^c dy\, dx$$

$$= 2\int_0^a \int_0^b \left(xc + yc + c^2/2\right) dy\, dx$$

$$= 2\int_0^a \left[xcy + \frac{y^2}{2}c + \frac{c^2}{2}y\right]_0^b dx$$

$$= 2\int_0^a \left[xcb + \frac{b^2}{2}c + \frac{c^2}{2}b\right] dx$$

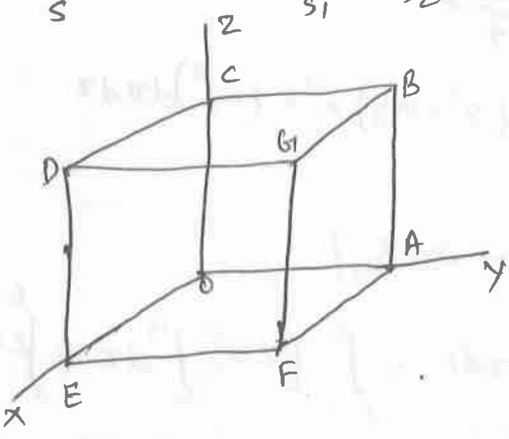
$$= 2\left[bc\frac{x^2}{2} + \frac{b^2c}{2}x + \frac{bc^2}{2}x\right]_0^a$$

$$= 2\left[\frac{bca^2}{2} + \frac{b^2ca}{2} + \frac{bc^2a}{2}\right]$$

$$= \cancel{2}\left(\frac{abc}{\cancel{2}}\right)[a+b+c]$$

$$= abc\,[a+b+c]$$

$$L.H.S = \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



	Face	unit outward normal vectors $(\hat{n})$	
S_1	OCDE	$\vec{i}$	$x=a$
S_2	ABGF	$-\vec{i}$	$x=0$
S_3	OCBA	$\vec{j}$	$y=b$
S_4	GDEF	$-\vec{j}$	$y=0$
S_5	CDGB	$\vec{k}$	$z=c$
S_6	OEFA	$-\vec{k}$	$z=0$

(i) $$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{OCDE} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(\vec{i})\,dy\,dz$$

$$= \int_0^c\int_0^b (x^2-yz)\,dy\,dz \qquad [x=a \text{ on } S_1]$$

$$= \int_0^c\int_0^b (a^2-yz)\,dy\,dz = \int_0^c \left[a^2y - \tfrac{y^2}{2}z\right]_0^b dz = \int_0^c \left[a^2b - \tfrac{b^2}{2}z\right]dz$$

$$= \left[a^2bz - \tfrac{b^2}{2}\tfrac{z^2}{2}\right]_0^c = \left[a^2bc - \tfrac{b^2}{2}\tfrac{c^2}{2}\right]$$

$$= a^2bc - \frac{b^2c^2}{4}$$

(ii) $$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{ABGF} (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\cdot(-\vec{i})\,dy\,dz$$

$$= -\int_0^c\int_0^b (x^2-yz)\,dy\,dz \qquad [x=0 \text{ on } S_2]$$

$$= -\int_0^c\int_0^b -yz\,dy\,dz = \int_0^c \left[\tfrac{y^2}{2}z\right]_0^b dz = \int_0^c \left(\tfrac{b^2}{2}z\right)dz$$

$$= \frac{b^2}{2}\int_0^c z\,dz = \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c = \frac{b^2}{2}\left[\frac{c^2}{2}\right] = \frac{b^2c^2}{4}$$

(iii) $$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{OCBA} (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\cdot(\vec{j})\,dx\,dz$$

$$= \int_0^c\int_0^a (y^2-zx)\,dx\,dz \qquad [y=b \text{ on } S_2]$$

$$= \int_0^c\int_0^a [b^2-zx]\,dx\,dz = \int_0^c \left[b^2x - z\tfrac{x^2}{2}\right]_0^a dz = \int_0^c \left[b^2a - z\tfrac{a^2}{2}\right]dz$$

$$= \left[b^2az - \tfrac{z^2}{2}\tfrac{a^2}{2}\right]_0^c = ab^2c - \frac{c^2a^2}{4} \;//$$

(iv) $$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{EDGF} (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\cdot(-\vec{j})\,dx\,dz$$

$$= -\int_0^c\int_0^a (y^2-zx)\,dx\,dz \qquad [y=0 \text{ on } S_4]$$

$$= -\int_0^c\int_0^a -zx\,dx\,dz = \int_0^c\int_0^a zx\,dx\,dz = \int_0^c \left[z\tfrac{x^2}{2}\right]_0^a dz = \int_0^c z\tfrac{a^2}{2}\,dz$$

$$= \frac{a^2}{2}\int_0^c z\,dz = \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^c = \frac{a^2}{2}\frac{c^2}{2} = \frac{a^2c^2}{4}$$

21

(v) $\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{CDGB} (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\cdot(\vec{k})\,dx\,dy$

$= \int_0^b \int_0^a (z^2-xy)\,dx\,dy \qquad [z=c \text{ on } S_5]$

$= \int_0^b \int_0^a (c^2-xy)\,dx\,dy = \int_0^b \left(c^2x - \frac{x^2}{2}y\right)_0^a dy = \int_0^b \left(c^2a - \frac{a^2}{2}y\right)dy$

$= \left[c^2ay - \frac{a^2}{2}\frac{y^2}{2}\right]_0^b = \left[c^2ab - \frac{a^2b^2}{4}\right] = abc^2 - \frac{a^2b^2}{4}$

(vi) $\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFA} (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\cdot(-\vec{k})\,dx\,dy$

$= -\int_0^b \int_0^a (z^2-xy)\,dx\,dy \qquad [z=0 \text{ on } S_6]$

$= -\int_0^b \int_0^a -xy\,dx\,dy = \int_0^b \int_0^a xy\,dx\,dy = \int_0^b \left(\frac{x^2}{2}y\right)_0^a dy$

$= \int_0^b \left[\frac{a^2}{2}y\right] dy = \left[\frac{a^2}{2}\frac{y^2}{2}\right]_0^b = \frac{a^2b^2}{4}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{c^2a^2}{4}} + \cancel{\frac{c^2a^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$

$= a^2bc + ab^2c + abc^2$

$= abc(a+b+c)$

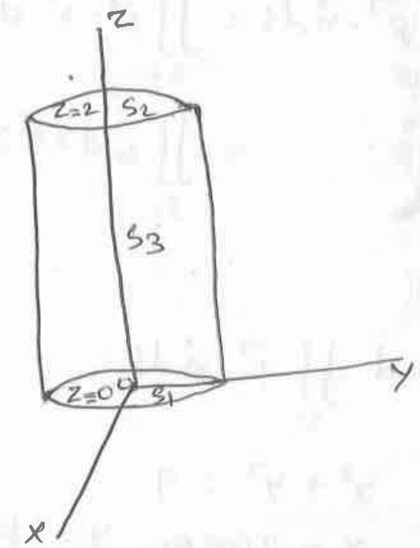
L.H.S = R.H.S

③ Verify Gauss divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cyclindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$

soln:

Gauss divergence theorem is

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$



Given $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$.

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$

$= 2z$

The region bounded by $x^2+y^2=9$

$z=0$ and $z=2$

$$R.H.S = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{2} 2z\, dz\, dy\, dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left[\frac{z^2}{2}\right]_0^2 dy\, dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\, dy\, dx$$

$$= 4\int_{-3}^{3} 2\,[y]_0^{\sqrt{9-x^2}}\, dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\, dx$$

$$= 16\int_{0}^{3}\sqrt{9-x^2}\, dx = 16\left[\frac{x}{3}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 = 16\left[\left(0+\frac{9}{2}\,\frac{\pi}{2}\right) - (0+0)\right]$$

$$= \cancel{16}^{\,4}\left[\frac{9\pi}{\cancel{4}}\right] = 36\pi$$

$$L.H.S = \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$\iint_{S_1}\vec{F}\cdot\hat{n}\, ds = \iint_{S_1}\vec{F}\cdot(-\vec{k})\, dx\, dy$$

$$= \iint_{S_1} -z^2\, dx\, dy \quad [z=0 \text{ on } s_1]$$

$$= 0$$

$$\iint_{S_2}\vec{F}\cdot\hat{n}\, ds = \iint_{S_2}\vec{F}\cdot\vec{k}\, dx\, dy = \iint_{S_2} z^2\, dx\, dy \quad [z=2 \text{ on } S_2]$$

$$= \iint_{S_2} 4\, dx\, dy = 4\iint_{S_2} dx\, dy = 4\,[\text{Area of circle}]$$

$$= 4\pi r^2$$

$$= 4\pi(3)^2 = 4\pi(9) = 36\pi$$

To find $\iint_{S_3}\vec{F}\cdot\hat{n}\, ds$

$x^2+y^2=9$

$x = 3\cos\theta \quad y = 3\sin\theta$

$ds = 3\, d\theta\, dz$ $\quad$ [$\because ds = z\, d\theta\, dz$]

Hint: The surface S_3 on z axis and the radius is 3]

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$\phi = x^2+y^2-9$

$\nabla\phi = 2x\vec{i}+2y\vec{j}$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2\sqrt{9} = 2\times 3 = 6$

$\hat{n} = \dfrac{2x\vec{i}+2y\vec{j}}{6} = \dfrac{2}{6}(x\vec{i}+y\vec{j}) = \frac{1}{3}(x\vec{i}+y\vec{j})$

$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{S_3} (y\vec{i}+x\vec{j}+z^2\vec{k})\cdot\frac{1}{\cancel{3}}(x\vec{i}+y\vec{j})\,\cancel{3}\, dz\, d\theta$

$= \iint_{S_3} (xy+xy)\, dz\, d\theta$

$= \iint_{S_3} 2xy\, dz\, d\theta = 2\int_0^{2\pi}\int_0^2 (3\cos\theta\, 3\sin\theta)\, dz\, d\theta$

$= 9\int_0^{2\pi}\int_0^2 2\sin\theta\cos\theta\, dz\, d\theta$

$= 9\int_0^{2\pi}\int_0^2 \sin 2\theta\, dz\, d\theta$

$= 9\int_0^{2\pi} \sin 2\theta \left[z\right]_0^2 d\theta = 9\int_0^{2\pi} \sin 2\theta\, [2-0]\, d\theta$

$= 18\int_0^{2\pi} \sin 2\theta\, d\theta = 18\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$

$= 9\left[-\cos 2\theta\right]_0^{2\pi} = -9\left[\cos 4\pi - \cos 0\right]$

$= -9[1-1] = -9(0) = 0$

$\therefore \iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$

$= 0 + 36\pi + 0$

$= 36\pi$

$\therefore$ L.H.S = R.H.S

④ If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, a, b, c are constants, show that $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{4\pi}{3}(a+b+c)$ where S is the surface of a unit sphere.

Proof:

W.K.T. G.D.T is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right)dV$$

$$= \iiint_V (a+b+c)\,dV$$

$$= (a+b+c)V$$

$$= (a+b+c)\tfrac{4}{3}\pi r^3$$

$$= (a+b+c)\tfrac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \tfrac{4}{3}\pi(a+b+c)$$

no need to solve

⑤ Using Divergence theorem of Gauss evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the sphere $x^2+y^2+z^2 = a^2$

Proof:

G.D.T. is $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$

$$\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

alternative method see book

$$= 3a^2$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 3a^2\,dV$$

$$= 3a^2 \iiint_V dV$$

$$= 3a^2 V = 3a^2 \tfrac{4\pi}{3} r^3$$

$$= \cancel{3}a^2 \frac{4\pi}{\cancel{3}} a^3$$

$$= 4\pi a^5$$

⑥ Evaluate using divergence theorem $\iint_S xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dx\,dy$ where S is the entire surface of the hemisphere region bounded by $x^2+y^2+z^2=a^2$ and $z=0$

Soln:

W.K.T. Cartesian form of Gauss divergence theorem

$$\iint_S xz^2\,dy\,dz + (x^2y-z^3)\,dz\,dx + (2xy+y^2z)\,dx\,dy = \iiint_V \left[\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2y-z^3) + \frac{\partial}{\partial z}(2xy+y^2z)\right] dx\,dy\,dz$$

$$= \iiint_V (x^2+y^2+z^2)\,dV = \iiint_V a^2\,dV = a^2 \tfrac{1}{2}V$$

$$= a^2 \times \frac{1}{\cancel{2}} \times \frac{\cancel{4}^{2}}{3}\pi a^3$$

$$= \frac{2}{3}\pi a^5$$

3.9 GRADIENT - DIRECTIONAL DERIVATIVE

Problems based on Gradient

① If $\phi = xyz$ find $\nabla\phi$

Soln

Given $\phi = xyz$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

$$= \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

② If $\phi = \log(x^2+y^2+z^2)$ find $\nabla\phi$

Given $\phi = \log(x^2+y^2+z^2)$

$$\left[\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \quad r^2 = x^2+y^2+z^2\right]$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\left[\frac{2x}{x^2+y^2+z^2}\right] + \vec{j}\left[\frac{2y}{x^2+y^2+z^2}\right] + \vec{k}\left[\frac{2z}{x^2+y^2+z^2}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\,\vec{r}$$

③ Find $\nabla(r)$

Soln

WKT $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2x = 2r\frac{\partial r}{\partial x} \qquad 2y = 2r\frac{\partial r}{\partial y} \qquad 2z = 2r\frac{\partial r}{\partial z}$$

$$\frac{\partial r}{\partial x} = x/r \,; \quad \frac{\partial r}{\partial y} = y/r \,; \quad \frac{\partial r}{\partial z} = z/r$$

$$\therefore \nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\, x/r + \vec{j}\, y/r + \vec{k}\, z/r$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = \vec{r}/r$$

④ prove that $\nabla(r^n) = n r^{n-2}\vec{r}$

Soln:

$$\nabla(r^n) = \Sigma \vec{i}\, \partial/\partial x\,(r^n) = \Sigma\vec{i}\; n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \Sigma\vec{i}\; n r^{n-1}\, x/r$$

$$= \Sigma\vec{i}\; n r^{n-2} x$$

$$= n r^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= n r^{n-2}\vec{r}.$$

⑤ Find $\nabla(\log r)$

$$\nabla(\log r) = \Sigma\vec{i}\,\partial/\partial x\,(\log r)$$

$$= \Sigma\vec{i}\; 1/r\; \frac{\partial r}{\partial x} = \Sigma\vec{i}\; 1/r\; x/r$$

$$= \Sigma\vec{i}\; 1/r^2\; x$$

$$= 1/r^2\,[x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= \vec{r}/r^2$$

⑥ prove that $\nabla(e^{x^2+y^2+z^2}) = 2e^{r^2}\vec{r}$

Soln:

WKT $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = x/r, \quad \frac{\partial r}{\partial y} = y/r, \quad \frac{\partial r}{\partial z} = z/r$$

$$\nabla\left(e^{x^2+y^2+z^2}\right) = \nabla(e^{r^2}) = \sum \vec{i}\, \partial/\partial x\,(e^{r^2})$$

$$= \sum \vec{i}\; e^{r^2}\, 2r \frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\; e^{r^2}\, 2\cancel{r}\; x/\cancel{r}$$

$$= \sum \vec{i}\; e^{r^2}\, 2x$$

$$= 2e^{r^2} \sum x\vec{i}$$

$$= 2e^{r^2}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]$$

$$= 2e^{r^2}\vec{r}$$

(4) prove that $\nabla f(r) = \dfrac{f'(r)}{r}\vec{r}$ where $r = x\vec{i}+y\vec{j}+z\vec{k}$

$$\nabla f(r) = \sum \vec{i}\frac{\partial}{\partial x} f(r)$$

$$= \sum \vec{i}\; f'(r)\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\; f'(r)\; x/r$$

$$= \frac{f'(r)}{r}\sum x\vec{i}$$

$$= \frac{f'(r)}{r}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]$$

$$= \frac{f'(r)}{r}\vec{r}$$

Directional Derivative

$$= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

(1) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of $2\vec{i}-\vec{j}-2\vec{k}$

Soln:

Given $\phi = x^2yz + 4xz^2$

$$\vec{a} = 2\vec{i}-\vec{j}-2\vec{k}$$

$$|\vec{a}| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla\phi = \vec{i}\,\partial/\partial x\,(x^2yz+4xz^2) + \vec{j}\,\partial/\partial y\,(x^2yz+4xz^2) + \vec{k}\,\partial/\partial z\,(x^2yz+4xz^2)$$

$$= (2xyz+4z^2)\vec{i} + x^2z\,\vec{j} + (x^2y+8xz)\vec{k}$$

$$\nabla\phi(1,-2,-1) = 8\vec{i}-\vec{j}-10\vec{k}$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (8\vec{i}-\vec{j}-10\vec{k}) \cdot \frac{2\vec{i}-\vec{j}-2\vec{k}}{3}$$

$$= \frac{1}{3}(8\vec{i}-\vec{j}-10\vec{k})\cdot(2\vec{i}-\vec{j}-2\vec{k})$$

$$= \frac{1}{3}[16+1+20]$$

$$= \frac{1}{3}[37] = \frac{37}{3}.$$

② Find the directional derivative of $\phi = x^2yz+4xz^2+xyz$ at $(1,2,3)$ in the direction of $2\vec{i}+\vec{j}-\vec{k}$

Soln:

$$\phi = x^2yz+4xz^2+xyz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$= (2xyz+4z^2+yz)\vec{i}+(x^2z+xz)\vec{j}+(x^2y+8xz+xy)\vec{k}$$

$$(\nabla\phi)(1,2,3) = 54\vec{i}+6\vec{j}+28\vec{k}$$

$$\vec{a} = 2\vec{i}+\vec{j}-\vec{k}$$

$$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

$$D.D = \nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|} = (54\vec{i}+65\vec{j}+28\vec{k})\cdot\frac{2\vec{i}+\vec{j}-\vec{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}[(54\vec{i}+6\vec{j}+28\vec{k})\cdot(2\vec{i}+\vec{j}-\vec{k})]$$

$$= \frac{1}{\sqrt{6}}[108+6-28] = \frac{1}{\sqrt{6}}[86] = \frac{86}{\sqrt{6}}$$

③ In what direction form the point $(2,1,-1)$ is the D. D of $\phi = x^3yz^3$ a maximum? What is the magnitude of this maximum

$$\phi = x^3yz^3$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$= 3x^2yz^3\vec{i}+x^3z^3\vec{j}+3x^3yz^2\vec{k}$$

$$(\nabla\phi)(2,1,-1) = -12\vec{i}-8\vec{j}+24\vec{k}$$

$$|\nabla\phi| = \sqrt{144+64+576} = \sqrt{784} = 28$$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = 28$.

<u>unit tangent vector</u>

$$\text{unit tangent vector} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

① Find a unit tangent vector to the following surfaces at the specified points $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$ at $t = 2$

Soln:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-6t)\vec{k}$$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}$$

$$\left[\frac{d\vec{r}}{dt}\right]_{t=2} = 4\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{16+16+4} = \sqrt{36} = 6$$

$$\text{unit tangent vector} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{4\vec{i}+4\vec{j}+2\vec{k}}{6}$$

$$= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3} \;//$$

NORMAL DERIVATIVE

Normal derivative $= |\nabla\phi|$

① Find the normal derivative of $\phi = xy+yz+zx$ at $(-1, 1, 1)$

Soln:

Given $\phi = xy+yz+zx$

$$\nabla\phi = \sum i \, \partial/\partial x \, (xy+yz+zx)$$

$$= (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

$$\nabla\phi\,(-1,1,1) = 2\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 2\vec{i}$$

Normal Derivative $|\nabla\phi| = \sqrt{4} = 2$

UNIT NORMAL VECTOR

$$\text{unit normal vector } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

① Find a unit normal to the surface $x^2y + 2xz^2 = 8$ at the point $(1, 0, 2)$

Soln:

Given $\phi = x^2y + 2xz^2 - 8$

$$\nabla\phi = (2xy+2z^2)\vec{i} + x^2\vec{j} + 4xz\,\vec{k}$$

$$\nabla\phi\,(1,0,2) = 8\vec{i} + \vec{j} + 8\vec{k}$$

$\therefore$ A unit normal to the given surface at the point is $\dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$$

② Find a unit vector normal to the surface $x^2+y^2-z=10$ at $(1,1,1)$

Soln

Given
$$\phi = x^2+y^2-z-10$$

$$\nabla\phi = 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\nabla\phi(1,1,1) = 2\vec{i}+2\vec{j}-\vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$$

ANGLE BETWEEN THE SURFACES

$$\cos\theta = \frac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

① Find the angle between the surfaces $z=x^2+y^2-3$ and $x^2+y^2+z^2=9$ at $(2,-1,2)$

Soln:

Given $\phi_1 = x^2+y^2-z-3$

$$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\nabla\phi_1(2,-1,2) = 4\vec{i}-2\vec{j}-\vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\phi_2 = x^2+y^2+z^2-9$$

$$\nabla\phi_2 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla\phi_2(2,-1,2) = 4\vec{i}-2\vec{j}+4\vec{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\cos\theta = \frac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|} = \frac{(4\vec{i}-2\vec{j}-\vec{k})\cdot(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}\ \sqrt{36}} = \frac{16+4-4}{\sqrt{21}\ 6} = \frac{\cancel{16}\,8}{\cancel{6}_3\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$ //.

③ Find the angle between the surfaces $x\log z = y^2-1$ and $x^2y = 2-z$ at the point $(1,1,1)$

Let $\phi_1 = y^2 - x\log z - 1$

$$\nabla\phi_1 = -\log z\,\vec{i}+2y\vec{j}-\frac{x}{z}\vec{k}$$

$$\nabla\phi_1(1,1,1) = 0\vec{i}+2\vec{j}-\vec{k}$$
$$= 2\vec{j}-\vec{k}$$

$$|\nabla\phi_1| = \sqrt{0+4+1} = \sqrt{5}$$

let $\phi_2 = x^2y - 2+z$

$\nabla\phi_2 = 2xy\vec{i} + x^2\vec{j} + \vec{k}$

$\nabla\phi_2(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$

$|\nabla\phi_2| = \sqrt{4+1+1} = \sqrt{6}$

$\therefore \cos\theta = \dfrac{(2\vec{j}-\vec{k})\cdot(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}\sqrt{6}} = \dfrac{0+2-1}{\sqrt{30}} = \dfrac{1}{\sqrt{30}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{30}}\right)$

SCALAR POTENTIAL ϕ

If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$. Then find the value of ϕ.

Given:-

$\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

$\vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$\dfrac{\partial\phi}{\partial x} = 2xyz \quad \dfrac{\partial\phi}{\partial y} = x^2z, \quad \dfrac{\partial\phi}{\partial z} = x^2y$

$\int\partial\phi = \int 2xyz\,\partial x$

$\phi_1 = 2yz\,\frac{x^2}{2} + f(y,z)$

$\phi_1 = x^2yz + f(y,z)$

$\int\partial\phi = \int x^2z\,\partial y$

$\phi_2 = x^2yz + f(x,z)$

$\int\partial\phi = \int x^2y\,\partial z$

$\phi_3 = x^2yz + f(x,y)$

$\phi = \phi_1 + \phi_2 + \phi_3 = x^2yz + \text{constant.}$

3.10 DIVERGENCE AND CURL - IRROTATIONAL AND SOLENOIDAL VECTOR FIELDS

DIVERGENCE AND CURL

$\text{DIV}\vec{F} = \nabla\cdot\vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ Here $\nabla\cdot\vec{F}$ is a Scalar quantity

$\text{curl}\,\vec{F} = \nabla\times\vec{F} = i\dfrac{\partial\vec{F}}{\partial x} + j\dfrac{\partial\vec{F}}{\partial y} + \vec{k}\dfrac{\partial\vec{F}}{\partial z}$

$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Problems:-

① If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ then find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$

Given $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\nabla\cdot\vec{F} = \partial/\partial x\,(x^2) + \partial/\partial y\,(y^2) + \partial/\partial z\,(z^2)$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[0-0] + \vec{k}\,[0-0]$$

$$= 0.$$

② If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$ find (i) $\nabla\cdot\vec{F}$, (ii) $\nabla(\nabla\cdot\vec{F})$, (iii) $\nabla\times\vec{F}$ (iv) $\nabla\cdot(\nabla\times\vec{F})$ and $\nabla\times(\nabla\times\vec{F})$ at the point $(1,1,1)$

Soln:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $$\nabla\cdot\vec{F} = \partial/\partial x\,(x^2 - y^2 + 2xz) + \partial/\partial y\,(xz - xy + yz) + \partial/\partial z\,(z^2 + x^2)$$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

(ii) $$\nabla(\nabla\cdot\vec{F}) = \partial/\partial x\,(x+5z)\vec{i} + \partial/\partial y\,(x+5z)\vec{j} + \partial/\partial z\,(x+5z)\vec{k}$$

$$= \vec{i} + 5\vec{k}$$

(iii) $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= -(x+y)\vec{i} - (2x - 2x)\vec{j} + (y+z)\vec{k}$$

$$= -(x+y)\vec{i} + (y+z)\vec{k}$$

(iv) $$\nabla\cdot(\nabla\times\vec{F}) = \partial/\partial x\,[-(x+y)] + \partial/\partial y\,(0) + \partial/\partial z\,(y+z)$$

$$= -1 + 0 + 1 = 0$$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -(x+y) & 0 & y+z \end{vmatrix}$

$= \vec{i} + \vec{k}$

$\nabla \cdot \vec{F}\,(1,1,1) = 6$

$[\nabla(\nabla \cdot \vec{F})](1,1,1) = \vec{i} + 5\vec{k}$

$\nabla \times \vec{F}\,(1,1,1) = -2\vec{i} + 2\vec{k}$

$[\nabla \cdot (\nabla \times \vec{F})]\,(1,1,1) = 0$

$[\nabla \times (\nabla \times \vec{F})]\,(1,1,1) = \vec{i} + \vec{k}$

③ Calculate the curl of the vector

$\vec{F} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\vec{k}$ at $(1,-1,1)$

Soln:- curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$

$= \vec{i}\,[-2yz - 0] - \vec{j}\,[z^2 - xy] + \vec{k}\,[6xy - xz]$

$= -2yz\,\vec{i} - \vec{j}\,[z^2 - xy] + \vec{k}\,[6xy - xz]$

$(\nabla \times \vec{F})\,(1,-1,1) = 2\vec{i} - \vec{j}\,[1+1] + \vec{k}\,[-6-1]$

$= 2\vec{i} - 2\vec{j} - 7\vec{k}$

SOLENOIDAL VECTOR, IRROTATIONAL VECTOR

solenoidal vector formula : $\nabla \cdot \vec{F} = 0$

Irrotational vector formula : $\nabla \times \vec{F} = 0$

problems:-

① Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal

Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

$\nabla \cdot \vec{F} = 0$

$\nabla \cdot \vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) \cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

Hence $\vec{F}$ is Solenoidal.

(2) If $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is Solenoidal find the value of λ.

Soln:-

Given $\nabla \cdot \vec{V} = 0$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$$1 + 1 + \lambda = 0$$

$$\boxed{\lambda = -2}$$

(3) Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational

Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = \sum \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right]$$

$$= \sum \vec{i}\,[x-x]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0.$$

(4) Find the Constants a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Soln:-

Given $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} i & j & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\vec{i}\,[c+1] - \vec{j}\,[4-a] + \vec{k}\,[b-2] = 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$c+1=0$ $\quad$ $4-a=0$ $\quad$ $b-2=0$

$\boxed{c=-1}$ $\quad$ $\boxed{a=4}$ $\quad$ $\boxed{b=2}$

⑤ If $\vec{A}$ is a constant vector, prove that curl $\vec{A} = 0$

Let $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[0-0] + \vec{k}\,[0-0] = 0$$

Hence curl $\vec{A} = 0$

⑥ If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ find curl curl $\vec{F}$.

Soln:

Given $\vec{F} = x^2y\,\vec{i} + y^2z\,\vec{j} + z^2x\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \vec{i}\,[0-y^2] - \vec{j}\,[z^2-0] + \vec{k}\,[0-x^2] = -y^2\vec{i} - z^2\vec{j} - x^2\vec{k}$$

$$\text{curl curl } \vec{F} = \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \vec{i}\,[0+2z] - \vec{j}\,[-2x-0] + \vec{k}\,[0+2y]$$

$$= 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$$

$$= 2[z\vec{i} + x\vec{j} + y\vec{k}]$$

LAPLACE OPERATOR

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace operator

$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ is called the Laplace equation.

① Find the value of $\nabla^2 \left(\frac{1}{x+y+z}\right)$

Soln:

$$\nabla^2 \left(\frac{1}{x+y+z}\right) = \Sigma \frac{\partial^2}{\partial x^2}\left(\frac{1}{x+y+z}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{-1}{(x+y+z)^2}\right) = \Sigma \frac{2}{(x+y+z)^3}$$

$$= \frac{6}{(x+y+z)^3}$$

(2) If $\phi = x^2 - y^2$ prove that $\nabla^2 \phi = 0$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\frac{\partial \phi}{\partial x} = 2x \quad \frac{\partial \phi}{\partial y} = -2y \quad \frac{\partial \phi}{\partial z} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 \quad \frac{\partial^2 \phi}{\partial y^2} = -2 \quad \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\nabla^2 \phi = 2 - 2 + 0 = 0$$

(3) Find $\nabla^2 (r^2)$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = x/r \quad \frac{\partial r}{\partial y} = y/r \quad \frac{\partial r}{\partial z} = z/r$$

$$\frac{\partial^2 r^2}{\partial x^2} = 2 \quad \frac{\partial^2 r^2}{\partial y^2} = 2 \quad \frac{\partial^2 r^2}{\partial z^2} = 2$$

$$\nabla^2 (r^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^2$$

$$= 2 + 2 + 2 = 6$$

(4) Show that $\nabla^2 (e^r) = e^r + \frac{2}{r} e^r$

$$\nabla^2 e^r = \Sigma \frac{\partial^2}{\partial x^2}(e^r) = \Sigma \frac{\partial}{\partial x}\left[e^r \frac{\partial r}{\partial x}\right]$$

$$= \Sigma \frac{\partial}{\partial x}\left[e^r \frac{x}{r}\right] = \Sigma \frac{\partial}{\partial x}\left[\underset{u}{e^r}\ \underset{v}{x}\ \underset{w}{\frac{1}{r}}\right]$$

$$= \Sigma \left[e^r x \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x} + e^r (1) \frac{1}{r} + e^r \frac{\partial r}{\partial x} x \frac{1}{r} \right]$$

$$= \Sigma \left[-e^r \frac{x}{r^2} \frac{x}{r} + \frac{e^r}{r} + e^r \frac{x}{r} \frac{x}{r} \right]$$

$$= \Sigma \left[-\frac{x^2}{r^3} e^r + \frac{e^r}{r} + \frac{x^2}{r^2} e^r \right]$$

$$= -\frac{(x^2+y^2+z^2)}{r^3} e^r + \frac{3e^r}{r} + \frac{(x^2+y^2+z^2)}{r^2} e^r$$

$$= -\frac{r^2}{r^3} e^r + \frac{3e^r}{r} + \frac{r^2}{r^2} e^r$$

$$= -\frac{e^r}{r} + \frac{3e^r}{r} + e^r = \frac{2e^r}{r} + e^r = \frac{2}{r} e^r + e^r$$

(5) prove that $\nabla^2 f(r) = f''(r) + \left(\frac{2}{r}\right) f'(r)$

Soln:

$$\nabla f(r) = \vec{i}\, f'(r)\frac{\partial r}{\partial x} + \vec{j}\, f'(r)\frac{\partial r}{\partial y} + \vec{k}\, f'(r)\frac{\partial r}{\partial z}$$

$$= \frac{f'(r)}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= f'(r)\frac{\vec{r}}{r}$$

Note $\nabla\cdot(\phi A) = \nabla\phi\cdot\vec{A} + \phi\nabla\cdot\vec{A}$

$$\nabla^2 f(r) = \nabla\cdot(\nabla f(r)) = \nabla\cdot\left(f'(r)\frac{\vec{r}}{r}\right) = \nabla f'(r)\cdot\left(\frac{\vec{r}}{r}\right) + f'(r)\,\nabla\cdot\frac{\vec{r}}{r}$$

$$= f''(r)\frac{\vec{r}}{r}\cdot\frac{\vec{r}}{r} + f'(r)\sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= f''(r) + f'(r)\sum \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2}$$

$$= f''(r) + f'(r)\sum \frac{r - \frac{x^2}{r}}{r^2}$$

$$= f''(r) + f'(r)\sum \frac{r^2 - x^2}{r^3}$$

$$= f''(r) + f'(r)\,\frac{3r^2 - [x^2+y^2+z^2]}{r^3}$$

$$= f''(r) + f'(r)\,\frac{3r^2 - r^2}{r^3}$$

$$= f''(r) + f'(r)\,\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\left(\frac{2}{r}\right)$$

3.11 Volume Integration

Line, Surface and Volume Integrals

(1) If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is curve in the xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$

Given $\vec{F} = 3xy\vec{i} - y^2\vec{j}$.

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2\,dy$$

$$y = 2x^2$$

$$dy = 4x\,dx.$$

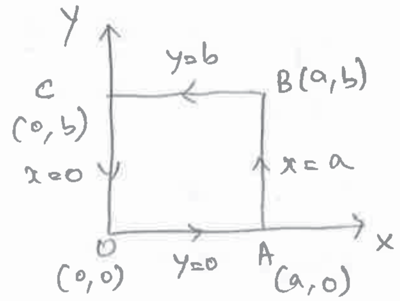
$$\therefore \vec{F}\cdot d\vec{r} = 3x(2x^2)\,dx - (2x^2)^2\,4x\,dx$$

$$= 6x^3\,dx - 16x^5\,dx$$
$$= (6x^3 - 16x^5)\,dx.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx$$
$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1$$
$$= \left(\frac{6}{4} - \frac{16}{6}\right) - (0-0)$$
$$= \left(\frac{3}{2} - \frac{8}{3}\right) = \frac{9-16}{6} = -\frac{7}{6}$$

(2) Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ the curve C is the Rectangle in the xy plane bounded by $x=0,\ x=a,\ y=b,\ y=0$

Soln:
Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$
$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$
$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$$



$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $\quad y=0 \quad dy=0$

AB $\quad x=a \quad dx=0$

BC $\quad y=b \quad dy=0$

CO $\quad x=0 \quad dx=0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} x^2\,dx + \int_{AB} -2ay\,dy + \int_{BC} (x^2+b^2)\,dx + \int_{CO}$$
$$= \int_0^a x^2\,dx - 2a\int_0^b y\,dy + \int_a^0 (x^2+b^2)\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0$$
$$= \left[\frac{a^3}{3} - 0\right] - 2a\left[\frac{b^2}{2} - 0\right] + \left[0 - \frac{a^3}{3} - ab^2\right]$$
$$= \cancel{\frac{a^3}{3}} - \cancel{2}\frac{ab^2}{\cancel{2}} - \cancel{\frac{a^3}{3}} - ab^2$$
$$= -ab^2 - ab^2$$
$$= -2ab^2$$

Surface integrals

(1) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is that part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Soln:

Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

Let $\phi = x^2+y^2+z^2-1$

$\nabla\phi = 2x\vec{i}+2y\vec{j}+2z\vec{k}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{1}} = x\vec{i}+y\vec{j}+z\vec{k}$$

$\vec{F}\cdot\hat{n} = xyz + xyz + xyz = 3xyz$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

Where R is the projection of S on the xy plane.

Clearly the projection R is bounded by the lines x axis $(y=0)$, y axis $(x=0)$ and the circle $x^2+y^2=1$, $z=0$

$|\hat{n}\cdot\vec{k}| = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k} = z$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R 3xyz\,\frac{dx\,dy}{z}$$

$$= 3\iint xy\,dx\,dy = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = 3\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= 3\int_0^1 \frac{x(1-x^2)}{2}\,dx$$

$$= 3/2\int_0^1 (x-x^3)\,dx$$

$$= 3/2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= 3/2\left[\tfrac{1}{2}-\tfrac{1}{4}\right] = 3/2\left[\tfrac{1}{4}\right] = 3/8 \text{ sq. units}$$

② Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=1$ included in the first octant between the planes $z=0$ and $z=2$.

<u>Soln:</u> Given $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$

$$\phi = x^2+y^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{x^2+y^2} = 2\sqrt{1} = 2$$

The unit normal $\hat{n}$ to the surface $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\vec{i}+2y\vec{j}}{2} = x\vec{i}+y\vec{j}$

$$\vec{F}\cdot\hat{n} = (z\vec{i}+x\vec{j}-y^2z\vec{k})\cdot(x\vec{i}+y\vec{j}) = xz+xy$$

Now $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

Where R is the projection of S on yz plane.

$$[\hat{n}\cdot\vec{i} = (x\vec{i}+y\vec{j})\cdot\vec{i} = x]$$

$$= \iint_R (xy+xz)\frac{dy\,dz}{x}$$

$$= \iint_R (y+z)\,dy\,dz$$

$$= \int_0^2\int_0^1 (y+z)\,dy\,dz$$

$$= \int_0^2 \left(\frac{y^2}{2}+yz\right)_0^1 dz$$

$$= \int_0^2 \left(\frac{1}{2}+z\right)dz = \left[\frac{z^2}{2}+\frac{1}{2}z\right]_0^2$$

$$= \left(\frac{4}{2}+\frac{2}{2}\right) = (2+1) = 3$$

$y = 1-x^2$

$= 1$

y

Volume integrals

① If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ evaluate $\iiint_V \nabla\times\vec{F}\,dV$ where V is the region bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$

<u>Soln:</u>

$$\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}\,(0) + \vec{j}\,(-3+4) + \vec{k}\,(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\therefore \iiint_V \nabla \times \vec{F}\, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[z\vec{j} - 2yz\vec{k} \right]_0^{4-2x-2y} dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[(4-2x-2y)\vec{j} - 2y(4-2x-2y)\vec{k} \right] dy\, dx.$$

$$= \int_0^2 \int_0^{2-x} \left[(4-2x-2y)\vec{j} - (8y - 4xy - 4y^2)\vec{k} \right] dy\, dx$$

$$= \int_0^2 \left[\left(4y - 2xy - 2\frac{y^2}{2}\right)\vec{j} - \left(8\frac{y^2}{2} - 4x\frac{y^2}{2} - 4\frac{y^3}{3}\right)\vec{k} \right]_0^{2-x} dx$$

$$= \int_0^2 \left[\left(4(2-x) - 2x(2-x) - \cancel{2}\frac{(2-x)^2}{\cancel{2}}\right)\vec{j} - \frac{8}{2}(2-x)^2 - \frac{4x}{2}(2-x)^2 - \frac{4}{3}(2-x)^3 \vec{k} \right] dx$$

$$= \int_0^2 \left[8 - 4x - 4x + 2x^2 - (4 + x^2 - 4x)\vec{j} - 4(4 + x^2 - 4x) - 2x(4 + x^2 - 4x) - \frac{4}{3}(8 - 12x + 6x^2 - x^3)\vec{k} \right] dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3) \right] dx$$

$$= \left[\left(4x - 4\frac{x^2}{2} + \frac{x^3}{3}\right)\vec{j} - \frac{\vec{k}}{3}\left(16x - 24\frac{x^2}{2} + 12\frac{x^3}{3} - 2\frac{x^4}{4}\right) \right]_0^2$$

$$= \left[\left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8) \right]$$

$$= \frac{8}{3}\vec{j} - \frac{\vec{k}}{3}(64 - 56)$$

$$= \frac{8}{3}\vec{j} - \frac{\vec{k}}{3}(8)$$

$$= \frac{8}{3}\vec{j} - \frac{8}{3}\vec{k}$$

$$= \frac{8}{3}(\vec{j} - \vec{k})$$

ANALYTIC FUNCTIONS

Continuous function:-

A function $f(z)$ is said to be continuous at $z = z_0$ if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Differentiable function

A function $f(z)$ is said to be differentiable at a point $z = z_0$ if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \text{ exists}$$

This limit is called the derivative of $f(z)$ at the point $z = z_0$

Note:- If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

Analytic function

A single-valued complex function $f(z)$ is said to be analytic at a point z_0 if it has a unique derivative at z_0. $f(z)$ is said to be analytic throughout a region R if it has a derivative at every point of R. (OR)

Entire function

A function which is analytic everywhere in the finite complex plane is called an entire function.

exs

The necessary condition for $f(z)$ to be analytic

Cauchy - Riemann Equations: (C-R equations)

The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(ie) $\boxed{u_x = v_y}$ $\boxed{v_x = -u_y}$

① Examine the following functions are analytic or not.

① $f(z) = e^x[\cos y + i\sin y]$

② $f(z) = \bar{z}$

③ $f(z) = z^3 + z$

④ $w = \sin z$

⑤ $f(z) = (x^2 - y^2 + 2xy) + i(x^2 - y^2 - 2xy)$

⑥ $f(z) = z^n$ polar.

⑦ $f(z) = \dfrac{x - iy}{2y^2}$

⑧ $f(z) = \dfrac{x - iy}{x^2 + y^2}$

① $f(z) = e^x(\cos y + i\sin y)$

$u + iv = e^x\cos y + ie^x\sin y$

$u = e^x\cos y$ $\quad$ $v = e^x\sin y$

$u_x = e^x\cos y$ — ① $\quad$ $v_x = e^x\sin y$ — ③

$u_y = -e^x\sin y$ — ② $\quad$ $v_y = e^x\cos y$ — ④

From ① & ④ $\quad$ From ② & ③

$\boxed{u_x = v_y}$ $\quad$ $\boxed{v_x = -u_y}$

Hence C-R eqn is satisfied. $\therefore f(z)$ is an analytic function.

② $f(z) = \bar{z}$ $\quad$ $z = x+iy$, $\bar{z} = x - iy$

$u + iv = x - iy$

$u = x$ $\quad$ $v = -y$

$u_x = 1$ $\quad$ $v_x = 0$

$u_y = 0$ $\quad$ $v_y = -1$

$u_x \neq v_y$

Hence CR eqn are not satisfied.

$\therefore f(z)$ is not analytic.

③ $f(z) = z^3 + z$

$u + iv = (x+iy)^3 + (x+iy)$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 + x + iy$

$= x^3 + i\,3x^2y - 3xy^2 - iy^3 + x + iy$

$u + iv = (x^3 - 3xy^2 + x) + i\,(3x^2y - y^3 + y)$

$u = x^3 - 3xy^2 + x$ $\qquad$ $v = 3x^2y - y^3 + y$

$u_x = 3x^2 - 3y^2 + 1$ —① $\qquad$ $v_x = 6xy$ —③

$u_y = -6xy$ —② $\qquad$ $v_y = 3x^2 - 3y^2 + 1$ —④

From (1) & (4) $\qquad$ From (2) & (3)

$u_x = v_y$ $\qquad$ $u_y = -v_x$

∴ CR eqns are Satisfied

∴ $f(z)$ is an analytic function.

④ Let $f(z) = w = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$f(z) = \sin x \cosh y + i \cos x \sinh y$

∴ $u = \sin x \cosh y$ $\qquad$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ —① $\qquad$ $v_x = -\sin x \sinh y$ —③

$u_y = \sin x \sinh y$ —② $\qquad$ $v_y = \cos x \cosh y$ —④

From (1) & (4) $\qquad$ From (2) & (3)

∴ $u_x = v_y$ and $u_y = -v_x$

⑥ $f(z) = (x^2 - y^2 + 2xy) + i\,(x^2 - y^2 - 2xy)$

$u + iv = x^2 - y^2 + 2xy + i\,(x^2 - y^2 - 2xy)$

$u = x^2 - y^2 + 2xy$ $\qquad$ $v = x^2 - y^2 - 2xy$

$u_x = 2x + 2y$ —① $\qquad$ $v_x = 2x - 2y$ —③

$u_y = -2y + 2x$ —② $\qquad$ $v_y = -2y - 2x$ —④

$u_x \neq v_y$ and $u_y \neq v_x$

Hence CR eqns are not satisfied

∴ $f(z)$ is not analytic.

⑥ $f(z) = z^n$

$$z = re^{i\theta}$$

$$z^n = r^n e^{in\theta}$$

$$= r^n[\cos n\theta + i\sin n\theta]$$

$$u+iv = r^n \cos n\theta + i r^n \sin n\theta$$

$u = r^n \cos n\theta$ $\qquad$ $v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1}\cos n\theta$ — ① $\qquad$ $\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$ — ③

$\frac{\partial u}{\partial \theta} = -n r^n \sin n\theta$ — ② $\qquad$ $\frac{\partial v}{\partial \theta} = n r^n \cos n\theta$ — ④

From ① & ④ $\qquad$ From ⑤ & ③

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ $\qquad$ $\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$

Hence C-R equations are satisfied.

$\therefore f(z) = z^n$ is analytic.

⑦ $f(z) = \frac{x-iy}{2y^2}$

$$f(z) = \frac{x}{2y^2} - \frac{i}{2y}$$

$$u+iv = \frac{x}{2y^2} - \frac{i}{2y}$$

$u = \frac{x}{2y^2}$ $\qquad$ $v = -\frac{1}{2y}$

$u_x = \frac{1}{2y^2}$ — ① $\qquad$ $v_x = 0$ — ③

$u_y = \frac{x}{\cancel{2}}\left(\frac{-\cancel{2}}{y^3}\right)$ $\qquad$ $v_y = -\frac{1}{2}\left(\frac{-1}{y^2}\right)$

$u_y = \frac{-x}{y^3}$ — ② $\qquad$ $v_y = \frac{1}{2y^2}$ — ④

From (1) and (4) $\qquad$ From (2) & (3)

$u_x = v_y$ and $u_y \neq -v_x$

$\therefore$ CR eqns are not satisfied

Hence $f(z)$ is not analytic

⑧ $f(z) = \frac{x - iy}{x^2+y^2}$

$f(z) = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$ $v = \frac{-y}{x^2+y^2}$

$u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$ —①

$u_y = \frac{-x\,2y}{(x^2+y^2)}$ —②

$v = \frac{-y}{x^2+y^2}$

$v_x = \frac{y\,2x}{x^2+y^2}$ —(3)

$v_y = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2}$

$= \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ —④

From (1) and (4) From ② & ③

$u_x = v_y$ $u_y = -v_x$

CR eqns are satisfied. $\therefore f(z)$ is analytic.

Sufficient condition for f(z) to be analytic

If the partial derivaties are u_x, u_y, v_x, v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$. Then the function $f(z)$ is analytic in a domain D.

C-R equations in polar form

If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta}$ then

$u_r = \frac{1}{r} v_\theta$ $v_r = -\frac{1}{r} u_\theta$

(ie) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Problems:-

① Show that $f(z) = 1/z$ is analytic everywhere except at $z = 0$ and find $f'(z)$

Soln: let $z = re^{i\theta}$

$$f(z) = \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$f(z) = \frac{\cos\theta}{r} - \frac{i\sin\theta}{r}$$

$$u + iv = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

$$u = \frac{\cos\theta}{r} \qquad v = -\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial r} = -\frac{1}{r^2}\cos\theta \quad \text{—(1)} \qquad \frac{\partial v}{\partial r} = \frac{1}{r^2}\sin\theta \quad \text{—(3)}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sin\theta}{r} \quad \text{—(2)} \qquad \frac{\partial v}{\partial \theta} = -\frac{\cos\theta}{r} \quad \text{—(4)}$$

From (1) & (4) From (2) & (3)

$$\frac{\partial u}{\partial r} = +\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

CR equations are satisfied

$\therefore$ At $z=0$ $r=0$ and so $f(z)$ is not defined at $z=0$

Hence $f(z)$ is analytic everywhere except at $z=0$

$$f(z) = u + iv$$

Diff w.r.to 'r'

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{-\cos\theta + i\sin\theta}{r^2 e^{i\theta}}$$

$$= -\frac{e^{-i\theta}}{r^2 e^{i\theta}} = -\frac{1}{(re^{i\theta})^2} = \frac{-1}{z^2}$$

(3) P.T. $f(z) = \cosh z$ is an analytic function and find its derivatives

$$\boxed{\cosh\theta = \cos i\theta}$$

Soln:-

Given $f(z) = \cosh z$

$$= \cos iz$$

$$= \cos i(x+iy)$$

$$= \cos(ix - y)$$

$$= \cos ix \cos y + \sin ix \sin y$$

$$= \cosh x \cos y + \sinh x \sin y$$

$$u + iv = \cosh x \cos y + \sinh x \sin y$$

$u = \cosh x \cos y$

$u_x = \sinh x \cos y$ — (1)

$u_y = -\cosh x \sin y$ — (2)

$v = \sinh x \sin y$

$v_x = \cosh x \sin y$ — (3)

$v_y = \sinh x \cos y$ — (4)

From (1) & (4)

$u_x = v_y$

From (2) and (3)

$u_y = -v_x$

$\therefore$ CR eqns are satisfied

$\therefore f(z)$ is analytic.

$f'(z) = u_x + i v_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x+iy)$

$= \sinh z$

(4) If $f(z) = (x-y)^2 + 2i(x+y)$ S.T. the CR eqns are satisfied along the curve $x-y=1$

Soln:-

$f(z) = (x-y)^2 + 2i(x+y)$

$u+iv = (x-y)^2 + 2i(x+y)$

$u = (x-y)^2$

$u_x = 2(x-y)$

$u_y = -2(x-y)$

$v = 2(x+y)$

$v_x = 2(1) = 2$

$v_y = 2(1) = 2$

$u_x = v_y$

$2(x-y) = 2$

$x-y=1$

$-v_x = +u_y$

$-2 = -2(x-y)$

$x-y = 1$

$\therefore$ CR eqns are satisfied only if $x-y=1$

(5) Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable anywhere.

Soln:-

$f(z) = xy + iy$

$u+iv = xy + iy$

$u = xy \qquad v = y$

x and y are continuous everywhere and ~~consequently~~ hence $u = xy$ and $v = y$ are continuous everywhere. Thus $f(z)$ is continuous everywhere.

$u_x = y$ $\quad v_x = 0$

$\quad\quad\quad v_y = 1$

$u_y = x$

$\therefore u_x \neq v_y \quad u_y \neq -v_x.$

$\therefore$ CR equations are not Satisfied.

Hence $f(z)$ is not differentiable anywhere through it is continuous everywhere.

⑥ S.T. $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic at $z=0$

Soln:-

Let $z = x+iy \quad \bar{z} = x-iy$

$$f(z) = |z|^2 = z\bar{z} = (x+iy)(x-iy)$$

$$= x^2+y^2$$

$$= u+iv$$

$u = x^2+y^2 \quad v = 0$

$u_x = 2x \quad v_x = 0$

$u_y = 2y \quad v_y = 0$

So the equations $u_x = v_y$ and $u_y = v_x$ are not Satisfied everywhere except at $z=0$ So $f(z)$ is differentiable only at $z=0$

Now $u_x = 2x$ $u_y = 2y$ $v_x = 0$ $v_y = 0$ are Continuous everywhere and in particular at $(0,0)$

Hence the Sufficient Condition for differentiable are Satisfied by $f(z)$ at $z=0$. So $f(z)$ is differentiable at $z=0$ only and not analytic there.

⑦ Find the constant a, b, c if $f(z) = (x+ay) + i(bx+cy)$ is analytic function.

Soln:-

$$f(z) = u+iv = (x+ay) + i(bx+cy)$$

$u = x+ay \quad v = bx+cy \quad u_x = v_y \quad u_y = -v_x$

$u_x = 1 \quad v_x = b \quad 1 = c \quad a = -b$

$u_y = a \quad v_y = c \quad \therefore \boxed{a = -b, \ c = 1}$

Given $f(z)$ is analytic

$\therefore$ CR equations are Satisfied

4.2 Properties of analytic function and Harmonic function

Defn of Laplace equation

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is Know as Laplace equation in two dimension. It is denoted by $\nabla^2 \phi = 0$

Property 1:-

The real and imaginary parts of an analytic function $w = u + iv$ satisfies the Laplace equation in two dimensions. (ie)

$\nabla^2 u = 0$ and $\nabla^2 v = 0$

Proof:-

Let $f(z) = w = u + iv$ be analytic

To prove:- u and v satisfy the Laplace equation

To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Given: $f(z)$ is analytic

$\therefore$ u and v satisfy CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---} (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} (2)$$

Diff (1) p.w.r to 'x' we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---} (3)$$

Diff (2) p.w.r to 'y' we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad \text{---} (4)$$

(3) + (4) $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$

$\therefore$ u Satisfy Laplace equation.

Diff (1) p.w.r to y we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 V}{\partial y^2} \quad \text{---} \ (5)$$

Diff (2) p.w.r to x we get

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 V}{\partial x^2} \quad \text{---} \ (6)$$

$$(5)+(6) \quad \frac{\partial^2 u}{\partial x \partial y} \ast \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$\therefore$ V satisfies Laplace equation.

<u>Property:- 2</u>

If $w = u(x, y) + iv(x, y)$ is an analytic function the curves of the family $u(x, y) = a$ & the curves of the family $v(x, y) = b$ acts orthogonally where a and b are varying constants

(or)

when the function $f(z) = u + iv$ is analytic, show that $u =$ constant and $v =$ constant are orthogonal.

Proof:-

Given $f(z)$ is an analytic function

$\therefore$ By C-R equation

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Given: $u(x, y) = a$ and $v(x, y) = b$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0, \ \& \ \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}.$$

$$\frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\partial u / \partial y}{\partial u / \partial x} \text{ (by CR eqns)}$$

$$= m_2 \text{ (say)}$$

$\therefore$ product of the slopes $= m_1 m_2$

$$= \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}$$

$$= -1$$

Hence the two family of curves form an orthogonal system.

property :- 3

An analytic function with constant modulus is constant

Proof :-

let $f(z) = u+iv$ be analytic. By CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$|f(z)| = \sqrt{u^2+v^2} = c \neq 0$$

$$|f(z)|^2 = u^2+v^2 = c^2 \text{ (say)}$$

(ie) $u^2+v^2 = c^2$

diff p.w.r to x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{---} (1)$$

diff p.w.r to y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$$

$$v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0 \quad \text{---} (2) \text{ (by CR eqn)}$$

(1) $\times u$ + (2) $\times v \Rightarrow$

$$u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + v^2\frac{\partial u}{\partial x} - uv\frac{\partial v}{\partial x} = 0$$

$$u^2+v^2\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\frac{\partial u}{\partial x} = 0 \quad [\because u^2+v^2 \neq 0]$$

① × v − ② × u ⇒

$$uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} - \left[uv\frac{\partial u}{\partial x} - u^2\frac{\partial v}{\partial x}\right] = 0$$

$$uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} - uv\frac{\partial u}{\partial x} + u^2\frac{\partial v}{\partial x} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad [\because u^2+v^2 \neq 0]$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0$$

$$f'(z) = 0$$

$f(z) = c$ Constant

Property :- 4

An analytic function whose real part is constant must itself be a constant.

(or)

If $f(z)$ is analytic show that $f(z)$ is constant if real part $f(z)$ is constant.

Proof:- Let $f(z) = u + iv$ be an analytic function

$\therefore$ by CR eqn $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$

Given $u =$ Constant

To Prove :- $f(z)$ is a Constant

$$u = c$$

$$\frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0$$

by CR equations

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0$$

$$f'(z) = 0$$

$$f(z) = c$$

$\therefore f(z)$ is a Constant.

Property :- 5

If $f(z)$ and $\overline{f(z)}$ are analytic in a region D. Show that $f(z)$ is constant in that region D.

Proof :-

Let $f(z) = u(x,y) + i v(x,y)$

$\overline{f(z)} = u(x,y) - i v(x,y)$

$\overline{f(z)} = u(x,y) + i[-v(x,y)]$

Since $f(z)$ is analytic in D we get $u_x = v_y$ & $u_y = -v_x$ ——(1)

Since $\overline{f(z)}$ is analytic in D $u_x = -v_y$ & $u_y = v_x$ ——(2)

Adding (1) & (2)

$2u_x = 0 \qquad 2u_y = 0$

$u_x = 0 \qquad u_y = 0$

Hence $v_x = 0 \quad v_y = 0 \quad f'(z) = u_x + i v_x = 0 + i0 = 0$

$\therefore f(z)$ is a constant in D.

Book work

Show that $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z\, \partial \bar{z}}$

proof :-

Let f be a function of x and y where x and y are functions of z and $\bar{z}$

(ie) $x = \dfrac{z+\bar{z}}{2}, \quad y = \dfrac{z-\bar{z}}{2i}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x}\left(\tfrac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{1}{2i}\right)$$

$$\frac{\partial f}{\partial z} = \tfrac{1}{2}\left[\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right]$$

$$2\frac{\partial}{\partial z} = \left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right] \quad \text{——(1)}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\left(\tfrac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-1}{2i}\right) = \tfrac{1}{2}\left[\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right]$$

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}$$

$$2\frac{\partial}{\partial \bar{z}} = \left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right] \quad \text{——— (2)}$$

Multiply (1) & (2)

$$2\frac{\partial}{\partial z} \times 2\frac{\partial}{\partial \bar{z}} = \left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right]\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$\boxed{4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} \;//$$

Problems:- (x)(x)

(1) If $f(z)$ is analytic show that

(i) $\nabla^2 |f(z)|^2 = 4|f'(z)|^2$

(ii) $\nabla^2 \log|f(z)| = 0$

Soln:-

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

$$\nabla^2 = 4\frac{\partial^2}{\partial z\partial \bar{z}}$$

$$|f(z)|^2 = f(z)\,\overline{f(z)}$$

$$\nabla^2 |f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\,\overline{f(z)}\right] \qquad \left[\because z\bar{z} = |z|^2\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\,\overline{f'(z)}\right]$$

$$= 4\left[f'(z)\,\overline{f'(z)}\right]$$

$$= 4|f'(z)|^2 \;//$$

(ii) $\nabla^2 \log|f(z)| = 0$

$$|f(z)|^2 = f(z)\,\overline{f(z)}$$

$$|f(z)| = \sqrt{f(z)\,\overline{f(z)}}$$

$\log(mn)^{1/2} = \frac{1}{2}\log mn$

$= \frac{1}{2}[\log m + \log n]$

$$\log|f(z)| = \log\sqrt{f(z)\,\overline{f(z)}}$$

$$= \frac{1}{2}\left[\log f(z) + \log \overline{f(z)}\right]$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 4\frac{\partial^2}{\partial z\,\partial\bar z}\,\tfrac{1}{2}\left[\log f(z)+\log f(\bar z)\right]$$

$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\left[\log f(z)+\log f(\bar z)\right]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar z}\log f(z)\right]+2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar z}\log f(\bar z)\right]$$

$$= 2\frac{\partial}{\partial z}[0]+2\frac{\partial}{\partial z}\left[\frac{1}{f(\bar z)}\,\overline{f'(z)}\right]$$

$$= 0+2\left[\frac{\partial}{\partial z}\frac{\overline{f'(z)}}{f(\bar z)}\right] = 2[0]=0$$

(2) If $f(z)=u+iv$ is a rectangular function of z in a domain D. prove that.

(i) $\nabla^2\left[|f(z)|\right]^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2$

(ii) $\nabla^2 u^p = p(p-1)\,u^{p-2}\,|f'(z)|^2$

(iii) $\nabla^2[\arg f(z)] = 0$

(iv) $\nabla^2|\mathrm{Re}\, f(z)|^2 = 2|f'(z)|^2$

(v) $\nabla^2|\mathrm{Im}\, f(z)|^2 = 2|f'(z)|^2$

(vi) $\left[\partial/\partial x\,|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$

(i) Soln:

$$\nabla^2\left[|f(z)|\right]^p = \nabla^2\left[|f(z)|^2\right]^{p/2}$$

$$= \nabla^2\left[f(z)\;f(\bar z)\right]^{p/2}$$

$$= \nabla^2\left[f(z)^{p/2}\,(f(\bar z))^{p/2}\right]$$

$$= \frac{4\partial^2}{\partial z\,\partial\bar z}\left[f(z)^{p/2}\,(f(\bar z))^{p/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[(f(z))^{p/2}\;\partial/\partial\bar z\,\left[f(\bar z)\right)^{p/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[(f(z))^{p/2}\;p/2\;f(\bar z)^{p/2-1}\;f'(\bar z)\right]$$

$$= 4\frac{\partial}{\partial z}\left[(f(z))^{p/2}\;p/2\;f(\bar z)^{p/2-1}\;f'(\bar z)\right]$$

$$= 4\left[p/2\;f(z)^{p/2-1}\,f'(z)\;p/2\;f(\bar z)^{p/2-1}\;f'(\bar z)\right]$$

$$= p^2\,f'(z)\,f'(\bar z)\,\left[f(z)\;f(\bar z)\right]^{p/2-1}$$

$$p^2\left[|f'(z)|^2\right]\left[|f(z)|^2\right]^{p/2-1}$$

$$= p^2|f'(z)|^2\left[|f(z)|^2\right]^{\frac{p-2}{2}}$$

$$= p^2|f'(z)|^2\,|f(z)|^{p-2}$$

$$= p^2|f(z)|^{p-2}|f'(z)|^2$$

Hence proved//.

(ii) $\nabla^2(u^p) = p(p-1)u^{p-2}|f'(z)|^2$

Let $f(z) = u+iv$ is an analytic function.

$u_x = v_y$ — ①

$u_y = -v_x$ — ② by CR equation

> harmonic function
> A real fn of two real variable x & y that possesses continuous 2nd order p.d. and that satisfies Laplace eqn is called a harmonic function.

$u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$

[$\because$ u and v are harmonic functions]

by ① & ②

$u_xv_x + u_yv_y = 0$

$f'(z) = u_x + iv_x$

$|f'(z)| = \sqrt{u_x^2+v_x^2}$

$|f'(z)|^2 = u_x^2+v_x^2$

$\frac{\partial}{\partial x}(u^p) = [pu^{p-1}u_x]$

$$\frac{\partial^2}{\partial x^2}(u^p) = \partial/\partial x\left[pu^{p-1}u_x\right]$$

$$= p\left[u^{p-1}u_{xx} + u_x(p-1)u^{p-2}u_x\right]$$

$$\frac{\partial^2}{\partial x^2}(u^p) = p\left[u^{p-1}u_{xx} + (p-1)u^{p-2}u_x^2\right]$$

Ily

$$\frac{\partial^2}{\partial y^2}(u^p) = p\left[u^{p-1}u_{yy} + (p-1)u^{p-2}u_y^2\right]$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p) = pu^{p-1}[u_{xx}+u_{yy}] + p(p-1)u^{p-2}[u_x^2+u_y^2]$$

$$= pu^{p-1}[0] + p(p-1)u^{p-2}|f'(z)|^2$$

$$\nabla^2 (u^p) = p(p-1)\, u^{p-2}\, |f'(z)|^2$$

iii) $\nabla^2 [\arg f(z)] = 0$

Soln:-

$$\arg f(z) = \tan^{-1}\left(\frac{v}{u}\right)$$

$$\frac{\partial}{\partial x}[\arg f(z)] = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{v}{u}\right)\right]$$

$$= \frac{1}{1+\left(\frac{v}{u}\right)^2}\left[\frac{u v_x - v u_x}{u^2}\right] = \frac{u^2}{u^2+v^2}\left[\frac{u v_x - v u_x}{u^2}\right]$$

$$= \frac{u v_x - v u_x}{u^2+v^2}$$

$$\frac{\partial^2}{\partial x^2}[\arg f(z)] = \frac{\partial}{\partial x}\left[\frac{u v_x - v u_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[u v_{xx} + v_x u_x - v u_{xx} - u_x v_x] - [u v_x - v u_x][2u u_x + 2v v_x]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u v_{xx} - v u_{xx}] - 2[u^2 u_x v_x + u v v_x^2 - u v u_x^2 - v^2 u_x v_x]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u v_{xx} - v u_{xx}] - 2[u_x v_x (u^2-v^2) + uv(v_x^2 - u_x^2)]}{(u^2+v^2)^2}$$

|||ly

$$\frac{\partial^2}{\partial y^2}[\arg f(z)] = \frac{(u^2+v^2)[u v_{yy} - v u_{yy}] - 2[u_y v_y (u^2-v^2) + uv(v_y^2 - u_y^2)]}{(u^2+v^2)^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\arg f(z)] = \frac{(u^2+v^2)[u(v_{xx}+v_{yy}) - v(u_{xx}+u_{yy})] - 2[(u^2-v^2)[u_x v_x + u_y v_y] + uv[(v_x^2+v_y^2) - (u_x^2+u_y^2)]]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(0) - v(0)] - 2[(u^2-v^2)(0) + uv[(v_x^2+v_y^2) - (u_x^2+u_y^2)]]}{(u^2+v^2)^2}$$

$$= \frac{-2[uv]}{(u^2+v^2)^2}(0) \quad [\because u_x = v_y;\ u_y = -v_x]$$

$$= 0$$

(iv) $\nabla^2 |Re\, f(z)|^2 = 2|f'(z)|^2$

Soln:-

Let $f(z) = u+iv$ is an analytic function

$Re\, f(z) = u$

$|Re\, f(z)|^2 = u^2$

$\left[\because f(z) = u+iv \right.$
$f'(z) = u_x + i v_x$
$\left. |f'(z)|^2 = u_x^2 + v_x^2 \right]$

$\frac{\partial}{\partial x}(u^2) = 2uu_x$

$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x)$

$= 2[uu_{xx} + u_x u_x] = 2[uu_{xx} + u_x^2]$

$|||^{ly}$ $\frac{\partial^2}{\partial y^2}(u^2) = 2[uu_{yy} + u_y^2]$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2[u(u_{xx}+u_{yy}) + u_x^2 + u_y^2]$

$= 2[u(0) + u_x^2 + v_x^2]$ by C R

$= 2|f'(z)|^2$

(v) $\nabla^2 |Im\, f(z)|^2 = 2|f'(z)|^2$

Soln:-

Let $f(z) = u+iv$

$Im\, f(z) = v$

$|Im\, f(z)|^2 = v^2$

$\frac{\partial}{\partial x}(v^2) = 2vv_x$

$\frac{\partial^2}{\partial x^2}(v^2) = \partial/\partial x\,(2vv_x)$

$= 2[vv_{xx} + v_x v_x] = 2[vv_{xx} + v_x^2]$

$|||^{ly}$ $\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Im\, f(z)|^2 = 2[v(v_{xx}+v_{yy}) + v_x^2 + v_y^2]$

$= 2[v(0) + v_x^2 + v_y^2]$ by C. R.

$= 2|f'(z)|^2$

(vi) $\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$

Soln:

$$f(z) = u + iv$$

$$|f(z)|^2 = u^2 + v^2$$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left[2uu_x + 2vv_x\right]$$

$$= \frac{uu_x + vv_x}{\sqrt{u^2+v^2}}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{(uu_x + vv_x)^2}{u^2+v^2}$$

III^ly

$$\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{(uu_y + vv_y)^2}{u^2+v^2}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{(uu_x + vv_x)^2 + (uu_y + vv_y)^2}{u^2+v^2}$$

$$= \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2+v^2}$$

$$= \frac{u^2[u_x^2 + u_y^2] + v^2[v_x^2 + v_y^2] + 2uv[u_xv_x + u_yv_y]}{u^2+v^2}$$

$$= \frac{u^2[|f'(z)|]^2 + v^2[|f'(z)|]^2 + 2uv[0]}{u^2+v^2}$$

$$= \frac{(u^2+v^2)\,|f'(z)|^2}{(u^2+v^2)} = |f'(z)|^2 \;//$$

4.3 CONSTRUCTION OF ANALYTIC FUNCTIONS

Milne Thomson method

when u is given $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

where $\phi_1(x,y) = \frac{\partial u}{\partial x}$

$\phi_2(x,y) = \frac{\partial u}{\partial y}$

when v is given $f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz$ where

$\psi_1(x,y) = \partial v/\partial y$

$\psi_2(x,y) = \partial v/\partial x$

Problems:-

① Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$

Soln:-

$$u = e^x \cos y$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -e^x \sin y$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

Put $x = z$ & $y = 0$ then

$$\phi_1(z,0) = e^z \qquad \phi_2(z,0) = 0$$

By milne thomson method

$$f(z) = \int e^z dz - i(0)$$

$$= e^z + c$$

② Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Soln:-

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy - 6y$$

Put $x = z$ and $y = 0$ then

$$\phi_1(z,0) = 3z^2 + 6z$$

$$\phi_2(z,0) = 0$$

By milne thomson method

$$f(z) = \int (3z^2 + 6z)\,dz - i(0)$$

$$= \cancel{3}\frac{z^3}{\cancel{3}} + \cancel{6}\frac{z^2}{\cancel{2}} + c$$

$$= z^3 + 3z^2 + c$$

③ Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln:-

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,[2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{0 - \sin 2x\,[2\sinh 2y]}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)\, 2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(z,0) = 0$$

By milne Thomson method

$$f(z) = \int -\operatorname{cosec}^2 z \, dz - i(0)$$

$$= \cot z + c \,//$$

(4) Find the regular function whose imaginary part is $e^{-x}[x\cos y + y\sin y]$

Soln:

$$V = e^{-x}[x\cos y + y\sin y]$$

$$\psi_2(x,y) = \frac{\partial v}{\partial x} = -e^{-x}[x\cos y + y\sin y] + e^{-x}[\cos y]$$

$$\psi_1(x,y) = \frac{\partial v}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y]$$

$$\psi_2(z,0) = -e^{-z}(z) + e^{-z}$$

$$= e^{-z}(1-z)$$

$$\psi_1(z,0) = e^{-z}(0+0+0)$$

$$= 0$$

By milne thomson method

$$f(z) = \int 0 + i\int e^{-z}(1-z)\,dz$$

$$= i\left[(1-z)\frac{e^{-z}}{(-1)} - \frac{(-1)\,e^{-z}}{(-1)^2}\right] + c$$

$$= i\left[-e^{-z} + ze^{-z} + e^{-z}\right] + c$$

$$= i\left[ze^{-z}\right] + c$$

$$= ize^{-z} + c$$

⑤ Find $w = u + iv$ is an analytic function where the imaginary part is $x^2 - y^2 + \dfrac{x}{x^2+y^2}$

Soln:

$$V = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\varphi_2(x,y) = \frac{\partial V}{\partial x} = 2x + \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$\varphi_1(x,y) = \frac{\partial V}{\partial y} = -2y + \frac{0 - x(2y)}{(x^2+y^2)^2}$$

$$\varphi_2(z,0) = 2z + \frac{z^2 - 2z^2}{(z^2)^2}$$

$$= 2z - \frac{z^2}{z^4} = 2z - \frac{1}{z^2}$$

$$\varphi_1(z,0) = 0$$

By milne thomson method

$$f(z) = 0 + i\int\left(2z - \frac{1}{z^2}\right)dz$$

$$= i\int\left(2z - z^{-2}\right)dz$$

$$= i\left[2\frac{z^2}{2} - \frac{z^{-2+1}}{-2+1}\right] + c$$

$$= i\left[z^2 - \frac{z^{-1}}{-1}\right] + c = i\left[z^2 + z^{-1}\right] + c = i\left[z^2 + \frac{1}{z}\right] + c$$

⑥ If $f(z) = u + iv$ is an analytic function and $u - v = e^x[\cos y - \sin y]$ find $f(z)$ in terms of z

Soln:

$$f(z) = u + iv \quad \text{—— ①}$$

$$if(z) = iu - v \quad \text{—— ②}$$

① + ② $(1+i)\,f(z) = (u-v) + i(u+v)$

$$F(z) = U + iV$$

Where $F(z) = (1+i)\,f(z)$

$U = u - v$

$V = u + v$

$U = u - v = e^x(\cos y - \sin y)$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$\phi_1(z,0) = e^z$

$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$

$\phi_2(z,0) = e^z[-1] = -e^{-z}$

By milne Thomson method we have

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^z dz - i\int -e^z dz$

$= e^z + i e^z + c_1$ $\boxed{\frac{c_1}{1+i} = c}$

$(1+i) f(z) = e^z(1+i) + c_1$

$f(z) = e^z + c$ //.

④ Find the analytic function $f(z) = u+iv$ if $u+v = \frac{x}{x^2+y^2}$ and $f(1) = 1$

Soln:

$u+v$ is an Im-part of $(1+i)f(z)$

$f(z) = u+iv$ —①

$i f(z) = iu - v$ —②

①+② $(1+i) f(z) = u - v + i(u+v)$

$F(z) = U + iV$

where $F(z) = (1+i) f(z)$

$U = u - v$

$V = u + v$

Given $V = u+v = \frac{x}{x^2+y^2}$

$\psi_2(x,y) = \frac{\partial V}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$

$\psi_1(x,y) = \frac{\partial V}{\partial y} = \frac{0 - 2yx}{(x^2+y^2)^2}$

$\psi_2(z,0) = \frac{z^2 - 2z^2}{(z^2)^2} = \frac{-z^2}{z^4} = -\frac{1}{z^2}$

$\psi_1(z,0) = 0$

By milne Thomson method

$$F(z) = \int \psi_1(z,0)\, dz + i \int \psi_2(z,0)\, dz$$

$$= \int 0 + i \int -\frac{1}{z^2}\, dz$$

$$= -i \int z^{-2}\, dz$$

$$= -i \left[\frac{z^{-2+1}}{-2+1} \right] + c$$

$$= i\,[z^{-1}] + c$$

$$= i\,[1/z] + c$$

$$(1+i)\, f(z) = i/z + c$$

$$f(z) = \frac{i}{z(1+i)} + \frac{c}{(1+i)} \quad \text{—①}$$

Given $f(1) = 1$

$$1 = \frac{i}{1+i} + \frac{c}{1+i}$$

$$1 = \frac{i+c}{1+i}$$

$$1+i = i+c$$

$$\boxed{c=1}$$

Sub in eqn ① $\boxed{c=1}$

$$f(z) = \frac{i}{z(1+i)} + \frac{1}{(1+i)}$$

$$= \frac{i(1-i)}{z(1+i)(1-i)} + \frac{1-i}{(1+i)(1-i)}$$

$$= \frac{i+1}{z(1+1)} + \frac{1-i}{2}$$

$$\therefore f(z) = \frac{i+1}{2z} + \frac{1-i}{2} \,//.$$

⑧ Find the analytic function for which $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ is the real part. Hence determine the analytic function $u+iv$ for which $u+v$ is the above function.

Soln: already we proved in problem (3)

Let $f(z) = u + iv$ — ①

$i f(z) = iu - v$ — ②

① + ②

$(1+i) f(z) = u - v + i(u+v)$

$F(z) = U + iV$

Given $V = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$\psi_2(x,y) = \dfrac{\partial V}{\partial x} = \dfrac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$\psi_1(x,y) = \dfrac{\partial V}{\partial y} = \dfrac{0 - \sin 2x \cdot 2\sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$\psi_2(z,0) = \dfrac{-2}{1-\cos 2z} = \dfrac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$

$\psi_1(z,0) = 0$

By Milne's Thomson method

$F(z) = \int \phi_1(z,0)\,dz + i \int \phi_2(z,0)\,dz$

$= 0 + i \int -\operatorname{cosec}^2 z\,dz.$

$= +i \int -\operatorname{cosec}^2 z\,dz$

$= i\cot z + c_1$

$(1+i) f(z) = i\cot z + c_1$

$\boxed{\text{where } c = \dfrac{c_1}{1+i}}$

$f(z) = \dfrac{i}{(1+i)}\cot z + \dfrac{c_1}{1+i}$

$= \dfrac{i(1-i)}{2}\cot z + c$

$= \dfrac{i+1}{2}\cot z + c$ //

(4) Find the analytic functions $f(z) = u + iv$ given that

(i) $-2v = e^x(\cos y - \sin y)$

(ii) $2u + v = e^x(\cos y - \sin y)$

(iii) $u - 2v = e^x(\cos y - \sin y)$

Soln:

Let $f(z) = u + iv$ — ①

$-i f(z) = -iu + v$ — ②

$-2 \times (2) \Rightarrow 2i\, f(z) = 2iu - 2v$

$2i f(z) = -2v + 2iu$

$F(z) = U + iV$ where

$F(z) = 2i\, f(z)$

$U = -2v$

$V = 2u$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$\phi_2(x,y) = \frac{\partial V}{\partial y} = e^x(-\sin y - \cos y)$

$\phi_1(z,0) = e^z$

$\phi_2(z,0) = e^z(-1) = -e^z$

By milne Thomson method.

$F(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$

$= \int e^z dz + i \int +e^z dz$

$2i f(z) = e^z + i e^z + c$

$2i\, f(z) = e^z(1+i) + c$

$f(z) = \frac{1+i}{2i} e^z + \frac{c}{2i}$

$f(z) = \frac{1-i}{2} e^z + c_1$

(ii) (1) $\times 2 \Rightarrow 2u + i2v = 2f(z)$ — (3)

$\Rightarrow v - iu = -if(z)$ — (4)

(3) + (4) $(2u+v) + i(2v-u) = (2-i) f(z)$

$U + iV = F(z)$

where $F(z) = (2-i)f(z)$, $U = 2u+v$, $V = 2v-u$

$U = 2u + v = e^x(\cos y - \sin y)$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$\phi_2(x,y) = \frac{\partial V}{\partial y} = e^x(-\sin y - \cos y)$

$\phi_1(z,0) = e^{z}$

$\phi_2(z,0) = -e^{-z}$

By milne's Thomson method

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^{z}\,dz - i\int -e^{+z}\,dz$

$= \int e^{z}\,dz + i\int e^{z}\,dz$

$= (1+i)\int e^{z}\,dz$

$(2-i)f(z) = (1+i)\,e^{z} + c$

$f(z) = \frac{1+i}{2-i}e^{z} + c$

$= \frac{(1+i)\times(2-i)}{(2)^2-(i)^2}e^{z} + c$

$= \frac{1+3i}{5}e^{z} + c$

(iii) ① $- 2\times$② we get

$(u-2v) + i(2u+v) = (1+2i)f(z)$

$F(z) = U + iV$

$F(z) = (1+2i)f(z) \quad U = u-2v, \quad V = 2u+v$

Given $U = u-2v = e^{x}(\cos y - \sin y)$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^{x}(\cos y - \sin y)$

$\phi_2(x,y) = \frac{\partial V}{\partial y} = e^{x}(-\sin y - \cos y)$

$\phi_1(z,0) = e^{z}$

$\phi_2(z,0) = -e^{z}$

By milne's Thomson method.

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^{z}\,dz - i\int -e^{z}\,dz$

$= \int e^{z}\,dz + i\int e^{z}\,dz$

$= (1+i)\int e^{z}\,dz$

$(1+2i)f(z) = (1+i)\,e^{z} + c$

$$f(z) = \frac{1+i}{1+2i} e^{z} + c$$

$$1+i-2i-2i^2$$
$$1-i+2$$

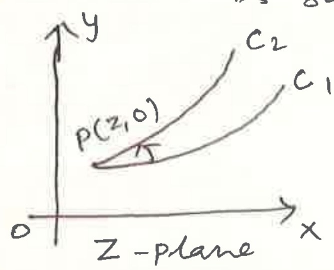
$$= \frac{(1+i)(1-2i)}{5} e^{z} + c$$

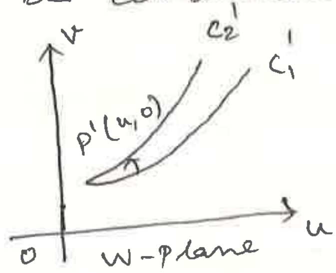
$$f(z) = \frac{3-i}{5} e^{z} + c$$ //

4.4 CONFORMAL MAPPINGS

Defn:-

A transformation that preserves angle between every pair of curves through a point both in magnitude and sense is said to be conformal mapping at that point.





ISOGONAL MAPPING

Defn:-

A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense are said to be isogonal at that point.

Fixed points (or) Invariant point

Fixed point of a mapping $w=f(z)$ are points that are mapped onto themselves. They are obtained from $w=f(z)=z$.

SOME STANDARD TRANSFORMATION

1. Translation

The translation $w=c+z$, where c is a complex constant, represents a translation.

Let $z=x+iy$ $\quad w=u+iv \quad c=a+ib$

$w=c+z$

$u+iv=a+ib+x+iy$

$u+iv=(x+a)+i(y+b)$

Equating the real and Imaginary parts.

$u = x + a \qquad v = y + b$

Hence the image of any point $P(x, y)$ in the z-plane is mapped onto the point $p'(x+a, y+b)$ in the w-plane.

① What is the region of w-plane into which the rectangular region in the z-plane bounded by the lines $x=0, y=0, x=1$ & $y=2$ is mapped under the transformation $w = z + (2-i)$

Soln:

$$w = z + (2-i)$$

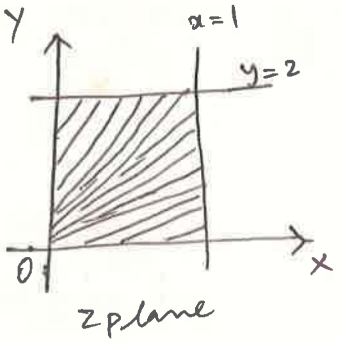
$$u + iv = x + iy + 2 - i$$

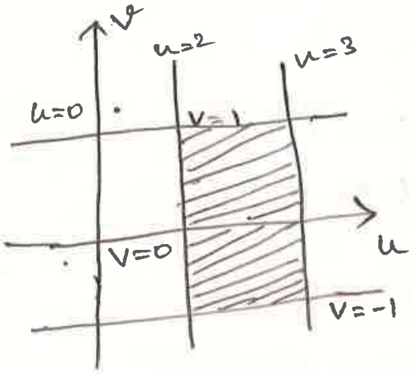
$$= (x+2) + i(y-1)$$

Equating the real and Imaginary parts

$u = x + 2 \qquad v = y - 1$

Given boundary lines are	Transformal boundary lines are
$x = 0$	$u = 0+2 = 2$
$y = 0$	$v = 0-1 = -1$
$x = 1$	$u = 1+2 = 3$
$y = 2$	$v = 2-1 = 1$





Hence the lines $x=0, y=0, x=1$ $y=2$ are mapped onto the lines $u=2, v=-1, u=3$ & $v=1$ respectively which forms a rectangle in w-plane.

② Find the image of circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Soln:-

$$u + iv = x + iy + 2 + 4i$$

$$= (x+2) + i(y+4)$$

Equating the real and Img. parts

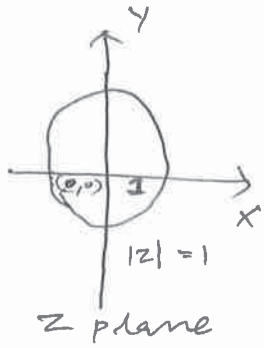
$u = x+2 \qquad v = y+4$

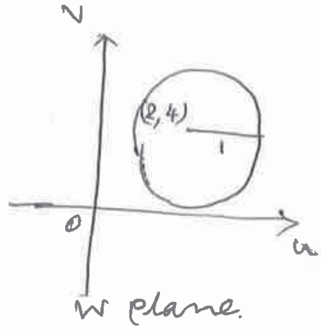
$\Rightarrow x = u-2 \qquad y = v-4$

Given $|z| = 1$

(ie) $x^2+y^2 = 1$

$(u-2)^2 + (v-4)^2 = 1$





Hence the circle $x^2+y^2=1$ is mapped into $(u-2)^2+(v-4)^2=1$ in w-plane which is also a circle centred at $(2,4)$ and radius 1.

(ii) MAGNIFICATION

$w = cz$ where c is a real constant represents magnification

$u+iv = c(x+iy)$

$= cx + icy$

$u = cx, \; v = cy$

$\therefore$ The image of the point (x, y) in the point (cx, cy)

MAGNIFICATION AND ROTATION

The transformation $w = cz$ where c is the complex constant represents both magnification and Rotation

Let $z = re^{i\theta}$ $\quad w = Re^{i\phi}$ $\quad c = a\,e^{i\alpha}$

$$Re^{i\phi} = (ae^{i\alpha})(re^{i\theta})$$

$$= ar\, e^{i(\alpha+\theta)}$$

$\therefore$ The transformation equation are $R = ar, \; \phi = \theta+\alpha$

Thus the point (r, θ) in the z-plane is mapped onto the point $(ar, \theta+\alpha)$ in the w-plane.

③ Determine the region D' of the w-plane into which the triangular region D enclosed by the lines $x=0, y=0, x+y=1$ is transformed under the transformation $w = 2z$

Soln:-

$w = 2z$

$u+iv = 2(x+iy)$

$u+iv = 2x+i2y$

$u = 2x \qquad v = 2y$

Given region D whose boundary lines are

$x = 0$

$y = 0$

$x + y = 1$

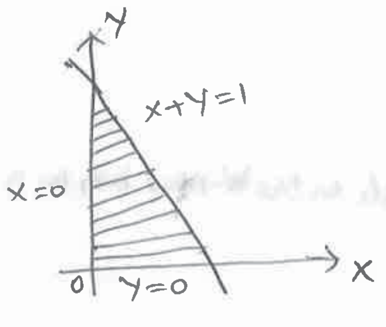
Transformed region D' whose boundary lines are

$u = 0$

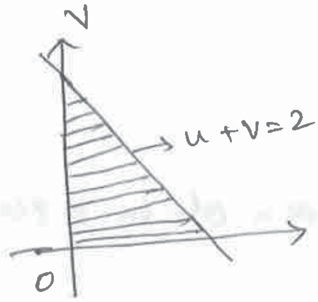
$v = 0$

$u/2 + v/2 = 1 \qquad [\because x = u/2,\ y = v/2]$

$\Rightarrow u + v = 2$



z plane



The line $x = 0$ in the z-plane is transformed to $u = 0$ in w-plane
The line $y = 0$ in the z-plane is transformed to $v = 0$ in w-plane
The line $x + y = 1$ in the z-plane is transformed to $u + v = 2$ in the w-plane.

4. Draw the image of a square whose vertices are (0,0) (1,0) (1,1) and (0,1) in the z-plane under the transformation $w = (1+i)z$

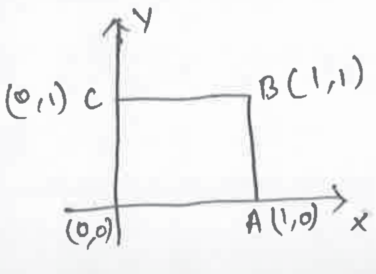
Soln:

$$u + iv = (1+i)(x+iy)$$
$$= x + iy + ix - y$$
$$= (x-y) + i(x+y)$$

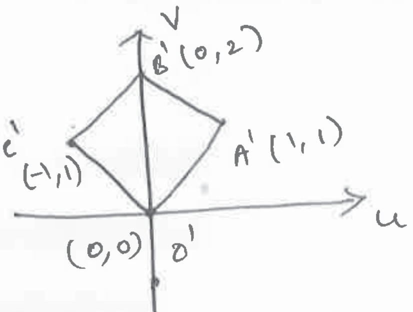
$u = x - y \qquad v = x + y$

Let OABC be a square in the z-plane

Z-plane	W-plane	
O (0,0)	$u=0 \quad v=0$	O' (0,0)
A (1,0)	$u=1 \quad v=1$	A' (1,1)
B (1,1)	$u=0 \quad v=2$	B' (0,2)
C (0,1)	$u=-1 \quad v=1$	C' (-1,1)



Thus we get the square $O'A'B'C'$ in the w-plane as shown in the figure below.



w-plane.

INVERSION AND REFLECTION

① S.T. the transformation $w = 1/z$ maps a circle in z plane into a circle in the w-plane or to a st. line.

The transformation $w = 1/z$ represents inversion w.r.to the unit circle $|z| = 1$ followed by reflection in the real axis.

$$w = 1/z \Rightarrow z = 1/w$$

$$x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad —① \qquad y = \frac{-v}{u^2+v^2} \quad —②$$

Consider the general equation of circle in z plane

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad —③$$

Sub ① & ② in ③

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$\frac{u^2+v^2+2gu(u^2+v^2)-2fv(u^2+v^2)+c(u^2+v^2)^2}{(u^2+v^2)^2} = 0$$

$$(u^2+v^2) + 2gu(u^2+v^2) - 2fv(u^2+v^2) + c(u^2+v^2)^2 = 0$$

$$(u^2+v^2)\left[1 + 2gu - 2fv + c[u^2+v^2]\right] = 0$$

$$c(u^2+v^2) + 2gu - 2fv + 1 = 0 \quad —④$$ which is a circle in the w-plane.

Hence under the transformation $w = 1/z$ a circle in z plane transforms to another circle in the w-plane. When the circle passes through the origin we have $c = 0$ in (3). Sub $c = 0$ in eqn (4) becomes

$29u - 26v + 1 = 0$ which is a straight line in w-plane.

① Find the image of modulus of $|z-2i| = 2$ under transformation $w = 1/z$

Soln:

$$w = 1/z \qquad z = 1/w$$

$$x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad \text{--- ①} \qquad y = \frac{-v}{u^2+v^2} \quad \text{--- ②}$$

Given

$$|z-2i| = 2$$

$$|x+iy-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$x^2+(y-2)^2 = 4$$

$$x^2+y^2-4y+4 = 4$$

$$x^2+y^2-4y = 0 \quad \text{--- ③}$$

Sub ① & ② in eqn ③

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{(u^2+v^2)} = 0$$

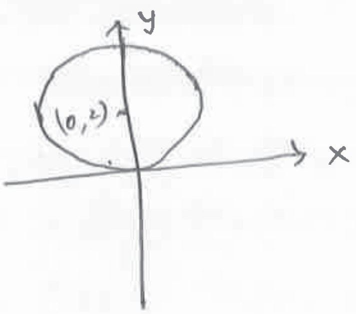
$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$(u^2+v^2)[1+4v] = 0$$

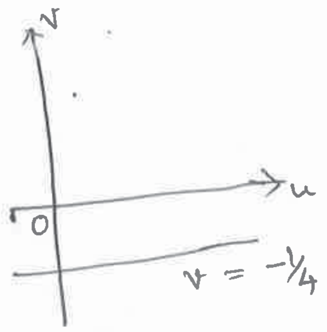
$$1+4v = 0 \qquad 4v = -1$$

$$v = -1/4$$

which is a st. line in the w-plane.



z-plane



w-plane

② Find the image of circle $|z-1|=1$ in the complex plane under the mapping $w = 1/z$

Soln

$$w = 1/z \qquad z = 1/w$$

$$x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \text{ --- ①} \qquad y = \frac{-v}{u^2+v^2} \text{ --- ②}$$

Given

$$|z-1| = 1$$

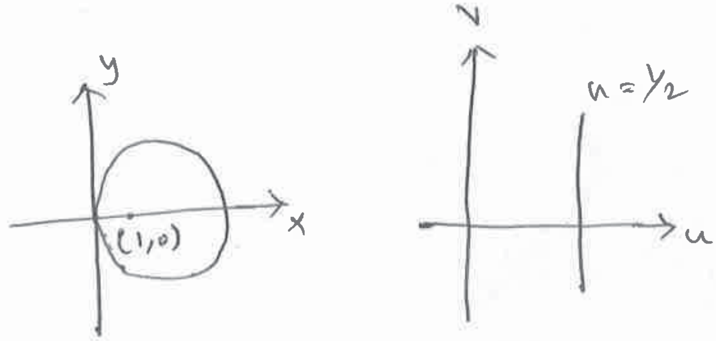
$$|x+iy-1| = 1$$

$$|(x-1)+iy| = 1$$

$$(x-1)^2 + y^2 = 1$$

$$x^2+1-2x+y^2 = 1$$

$$x^2+y^2-2x = 0 \text{ --- ③}$$



z - plane   w - plane

sub ① & ② in eqn ③

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - 2\left(\frac{u}{u^2+v^2}\right) = 0$$

$$\frac{u^2+v^2 - 2u(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$u^2+v^2[1-2u] = 0$$

$$1-2u = 0$$

$$1 = 2u$$

$\boxed{u = 1/2}$ which is a st. line in w-plane

③ Find the image of the infinite strips

(i) $1/4 < y < 1/2$

(ii) $0 < y < 1/2$ under the transformation $w = 1/z$

Soln:

$$w = 1/z \qquad z = 1/w$$

$$x = \frac{u}{u^2+v^2} \text{ --- ①} \qquad y = \frac{-v}{u^2+v^2} \text{ --- ②}$$

(i) Given $1/4 < y < 1/2$

when $y = 1/4$, $\dfrac{-v}{u^2+v^2} = 1/4$ by ②

$$u^2+v^2+4v = 0$$

$u^2 + (v+2)^2 = 4$ —— ③

This is a circle with centre at $(0,-2)$ and radius 2.

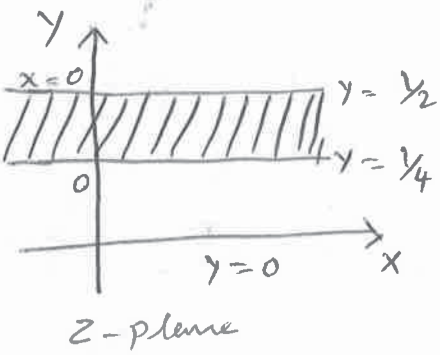
When $y = \frac{1}{2}$, $\frac{-v}{u^2+v^2} = \frac{1}{2}$

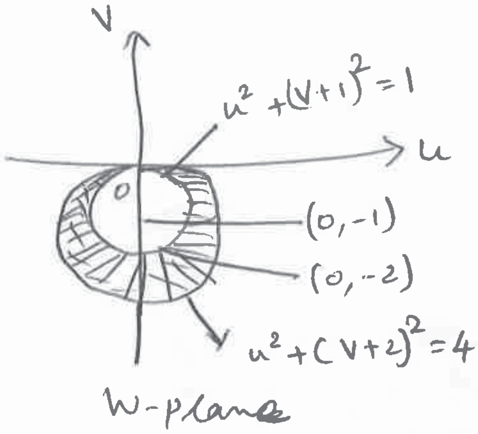
$u^2 + v^2 = -2v$

$u^2 + v^2 + 2v = 0$

$u^2 + (v+1)^2 = 1$ —— ④

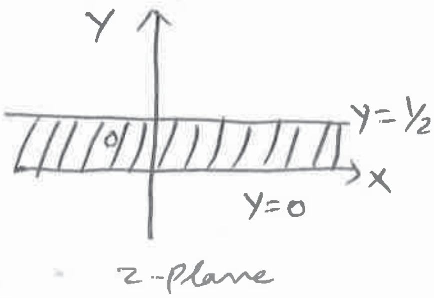
which is a circle centered at $(0,-1)$ and radius 1.

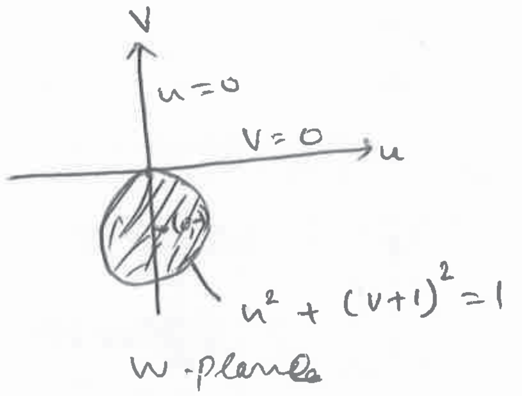




Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region inbetween circles. $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane

(ii) Given strip is $0 < y < \frac{1}{2}$





when $y = 0$

$v = 0$ by ②

when $y = \frac{1}{2}$ we get $u^2 + (v+1)^2 = 1$ by (4)

Hence the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w-plane

4. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$ and express them interm of w.

Soln:

Given $w^2 = (z-\alpha)(z-\beta)$ ——— (1)

Critical points occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$

diff (1) w.r.to z, we get

$$2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$2w\frac{dw}{dz} = 2z - (\alpha+\beta) \quad \text{——— (2)}$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z - (\alpha+\beta) = 0$$

$$2z = \alpha+\beta$$

$$z = \frac{\alpha+\beta}{2}$$

(2) $\Rightarrow$

$$2w\frac{dw}{dz} = \cancel{2}\left(\frac{\alpha+\beta}{\cancel{2}}\right)^{2z} - (\alpha+\beta)$$

$$w\frac{dw}{dz} = z - \frac{\alpha+\beta}{2}$$

$$\frac{dz}{dw} = \frac{w}{z-\frac{\alpha+\beta}{2}}$$

$$\frac{dz}{dw} = 0 \Rightarrow \frac{w}{z-\frac{(\alpha+\beta)}{2}} = 0$$

$$\Rightarrow w = 0 \Rightarrow (z-\alpha)(z-\beta) = 0$$

$\therefore z = \alpha, \beta$

$\therefore$ The critical points are $\frac{\alpha+\beta}{2}$, α and β

2. Find the critical point of the transformation $w = z + 1/z$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

$$\frac{dz}{dw} = \frac{z^2}{z^2-1}$$

$\boxed{\text{Critical points are } \pm 1, 0}$

Critical points occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$

$$\frac{z^2-1}{z^2} = 0 \Rightarrow z^2 - 1 = 0 \qquad z = \pm 1$$

$$\frac{z^2}{z^2-1} = 0 \qquad z^2 = 0 \qquad z = 0$$

(3) Find the point such that $w = f(z) = \sin z$ is not conformal

Soln The point at which $f(z)$ is not conformal are called critical points

Hence we required the critical point at $w = \sin z$

$\frac{dw}{dz} = \cos z$ $\qquad \frac{dz}{dw} = \frac{1}{\cos z}$

The C.P occur at $\frac{dw}{dz} = 0$ $\qquad \frac{dz}{dw} = 0$

$\cos z = 0$ $\qquad \frac{1}{\cos z} = 0 \Rightarrow 1 = 0$ which is Impossible

$z = \pm (2n-1)\pi/2 \qquad n = 0, \ldots\ldots$

4.5 BILINEAR TRANSFORMATION

The transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, d are complex numbers is called a bilinear transformation. It is also called mobius transformation and linear fractional transformation

The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Fixed points (or) invariant points

The fixed points of the transformation $w = \frac{az+b}{cz+d}$ is obtained from $z = \frac{az+b}{cz+d}$ (or) $cz^2 + (d-a)z - b = 0$

problems:-

(1) Find the fixed points of $w = \frac{2zi+5}{z-4i}$

The fixed points are given by

$z = \frac{2zi+5}{z-4i}$

$z(z-4i) = 2zi+5$

$z^2 - 4iz = 2zi+5$

$z^2 - 6zi - 5 = 0$

$z = \frac{6i \pm \sqrt{-36+20}}{2}$

$z = 5i, i$

② Find the invariant points of the bilinear transformation $\frac{z-1}{z+1}$

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z-1$$

$$z^2 + z = z-1$$

$$z^2 + z - z = -1$$

$$z^2 = -1$$

$$z = \pm\sqrt{-1}$$

$$z = i, -i$$

③ obtain the invariant points of the transformation $w = 2 - 2/z$

$$z = 2 - 2/z$$

$$z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2$$

$$z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

④ Determine the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in the w-plane.

Soln: Given $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

$w_1 = 0$, $w_2 = i$, $w_3 = 3i$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w}{-i} \quad \frac{-2i}{3i-w} = \frac{z+1}{-1} \quad \frac{-1}{1-z}$$

$$\frac{2w}{3i-w} = \frac{z+1}{1-z}$$

$$2w(1-z) = (z+1)(3i-w)$$

$$= 3zi + 3i - wz - w$$

$$2w(\cancel{1}-z) = 3i(z+1) - w(z+1)$$

$$2w(\cancel{1}-z)+w(z+1) = 3i(z+1)$$

$$w\left[2(\cancel{1}-z)+(z+1)\right] = 3i(z+1)$$

$$w\left[2-2z+z+1\right] = 3i(z+1)$$

$$w\left[3-z\right] = 3i(z+1)$$

$$w = \frac{3i(z+1)}{3-z} = -\frac{3i(z+1)}{z-3}$$ //

(5) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively.

Soln:- $z_1 = -2$, $z_2 = 0$, $z_3 = 2$

$w_1 = 0$, $w_2 = i$, $w_3 = -i$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i+i)}{(0-i)(-i-w)} = \frac{(z+2)(0-2)}{(-2-0)(2-z)}$$

$$\frac{w}{-i}\quad\frac{2i}{-(i+w)} = \frac{z+2}{-2}\quad\frac{-2}{2-z}$$

$$\frac{2w}{w+i} = \frac{z+2}{2-z}$$

$$\frac{w}{w+i} = \frac{1}{2}\left[\frac{z+2}{2-z}\right] = \frac{z+2}{4-2z}$$

$$\frac{w+(w+i)}{w-(w+i)} = \frac{(z+2)+(4-2z)}{(z+2)-(4-2z)}$$ by componendo and dividendo

$$\frac{2w+i}{-i} = \frac{-z+6}{3z-2}$$

$$2w+i = -i\left[\frac{-z+6}{3z-2}\right]$$

$$2w = -i\left[\frac{-z+6}{3z-2}\right]-i$$

$$w = \frac{-i}{2}\left[\frac{-z+6}{3z-2}+1\right]$$ //

⑥ Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Soln:-

Given $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ and $w_1 = 0$, $w_2 = i$, $w_3 = \infty$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\cancel{w_3}\left(1-\frac{w}{w_3}\right)} = \frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{\cancel{z_1}\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

$$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{(0-i)\left(1-\frac{w}{\infty}\right)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{\left(1-\frac{i}{\infty}\right)(0-z)}$$

$$\frac{w(-1)}{(-i)(1)} = \frac{(-1)(i)}{(1)(-z)}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = i\left(\frac{i}{z}\right) = \frac{i^2}{z} = \frac{-1}{z}$$

$$\boxed{w = \frac{-1}{z}}$$

⑦ Find the bilinear transformation that maps the points $1, 0, -1$ onto $\infty, -1, 0$ respectively.

Soln:-

Given $z_1 = 1$ $z_2 = 0$ $z_3 = -1$

$w_1 = \infty$ $w_2 = -1$ $w_3 = 0$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{\cancel{w_1}\left(\frac{w}{w_1}-1\right)(w_2-w_3)}{\cancel{w_1}\left(1-\frac{w_2}{w_1}\right)(w_3-w)} = \frac{(z-1)(0+1)}{(1-0)(-1-z)}$$

$$\frac{\left(\frac{w}{\infty}-1\right)(-1-0)}{\left(1-\frac{-1}{\infty}\right)(0-w)} = \frac{(z-1)(1)}{-1-z}$$

$$\frac{1}{-w} = \frac{z-1}{-1-z}$$

$$\frac{-1-z}{-w(z-1)} = 1$$

$$\frac{-1-z}{-z+1} = w$$

$$\boxed{w = \frac{-z-1}{-z+1}}$$

⑧ Find the bilinear transformation that maps the points $0, 1, \alpha$ and $i, 1, -i$ respectively.

Given

$$z_1 = 0 \quad z_2 = 1 \quad z_3 = \alpha$$

$$w_1 = i \quad w_2 = 1 \quad w_3 = -i$$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = \frac{(z-z_1)\cancel{z_3}\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)\cancel{z_3}\left(1-z/z_3\right)}$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = \frac{(z-0)(1/\alpha-1)}{(0-1)(1-z/\alpha)}$$

$$\frac{(w-i)(i+1)}{(i-1)(-i-w)} = \frac{-z}{-1} \Rightarrow z = \frac{w-i+wi-i^2}{-i^2+i-wi+w}$$

$$\frac{w-i}{(-i-w)} = z \times \frac{i-1}{1+i}.$$

$$w = \frac{z+i}{1+iz}$$

$$(w-i)(1+i) = z(i-1)(-i-w)$$

$$w+wi-i+1 = z[1-iw+i+w]$$

$$= z - ziw + zi + zw$$

$$w + wi + zi\bar{w} - zw = i - 1 + z + zi$$

$$w(1-z) + wi(1+z) = (1-i) + z(1+i)$$

$$w = \frac{(1-i) + z(1+i)}{1-z + i(1+z)}$$

COMPLEX INTEGRATION

Cauchy's Theorem (or) Cauchy's Integral theorem

Cauchy's Integral formula for derivatives

$$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} \, dz$$

$$\therefore f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} \, dz \quad \text{(or)} \quad \int_c \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a)$$

$$f'(a) = \frac{1!}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} \, dz \quad \text{(or)} \quad \int_c \frac{f(z)}{(z-a)^2} dz = 2\pi i\, f'(a)$$

$$f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)\,dz}{(z-a)^3} \quad \text{(or)} \quad \frac{2\pi i}{2!} f''(a) = \int_c \frac{f(z)\,dz}{(z-a)^3}$$

① State Cauchy's integral theorem

If a function $f(z)$ is analytic and its derivatives $f'(z)$ is continuous at all points inside and on a simple closed curve c, then $\int_c f(z)\, dz = 0$

② Evaluate $\int_c \frac{z\,dz}{z-2}$ where c is the circle $|z| = 1$

Soln:

we know that Cauchy's integral formula is

$$\int_c \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a) \quad [a \text{ lies inside } c]$$

Here $f(z) = z$

$a = 2$

$f(a) = a = 2$

Here $a=2$ lies out side the circle $|z| = 1$

$\because |z| = 1$

$x^2+y^2 = 1$

$\therefore$ By cauchy's integral formula $\int_C \frac{z\,dz}{z-2} = 0$

Note:-

$f(z)$ is analytic inside and on a closed curve c of a simply connected region R and if 'a' is any point with in c.

(3) Evaluate $\int_c \frac{1}{2z-3} dz$ if c is the circle $|z|=1$

Soln:-

we know that cauchy's integral formula is

$$\int_c \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a) \quad [a \text{ lies inside } c]$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \text{if } a \text{ lies inside } c$$

$$= 0 \quad \text{if } a \text{ lies outside } c$$

Given $\int_C \frac{1}{2z-3}\,dz = \frac{1}{2}\int_C \frac{dz}{z-3/2}$

Here $f(z) = 1$

$a = 3/2 = 1.5$ lies out side the circle $|z|=1$

Hence by cauchy's integral formula

$$\int_C \frac{1}{2z-3}\,dz = 2\pi i(0) = 0$$ [$\because$ $f(z)$ is not analytic outside C, $f(z)$ is analytic inside and on C]

(3) Evaluate $\int_C \frac{3z^2+7z+1}{z+1}\,dz$ where c is $|z| = 1/2$

soln:

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Given $\int_C \frac{3z^2+7z+1}{z+1}\,dz = \int_C \frac{3z^2+7z+1}{z-(-1)}\,dz$

Here $f(z) = 3z^2+7z+1$

$a = -1$ lies out side $|z| = 1/2$

$\therefore$ by Cauchy's Integral formula

$$\int_C \frac{3z^2+7z+1}{z+1}\,dz = 0$$

(4) Evaluate $\int_C \frac{1}{2z+3}\,dz$ where c is $|z|=2$

$$\frac{1}{2}\int_C \frac{1}{z-(-3/2)}\,dz$$

$a = -3/2 = -1.5$ is lies inside $|z|=2$

$$\frac{1}{2}\int_C \frac{1}{(z+3/2)}\,dz = \frac{1}{2}\, 2\pi i\, f(-3/2)$$

$$= \frac{1}{2}\, 2\pi i\,(1) = \pi i \qquad [\because f(z) = 1]$$

⑤ Evaluate $\int_C \frac{dz}{ze^z}$ where c is $|z|=1$

Soln:-

WKT Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Given $\int_C \frac{dz}{ze^z} = \int_C \frac{e^{-z}\,dz}{z-0}$

Here $f(z) = e^{-z}$

$a=0$ is lies inside $|z|=1$

By Cauchy's integral formula we get

$$\int_C \frac{e^{-z}\,dz}{z} = 2\pi i\, f(0) \qquad \left[\because f(z) = e^{-z}\right.$$

$$= 2\pi i\,(1) \qquad \left. f(0) = e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1\right]$$

$$= 2\pi i$$

⑥ Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where c is $|z|=2$ using Cauchy's integral formula.

Soln:-

WKT Cauchy's integral formula is

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

Given $\int_C \frac{z\,dz}{(z-1)^3}$ Here $f(z)=z$

$a=1$ is lies inside $|z|=2$

By Cauchy's integral formula we get

$$\int_C \frac{z\,dz}{(z-1)^3} = \frac{2\pi i}{2!} f''(1) \qquad \left[\because f(z)=z \quad f'(z)=1\right.$$

$$= \pi i\,(0) \qquad \left. f''(z)=0 \quad f''(1)=0\right]$$

$$=0$$

⑦ Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ if C is $|z|=1$

Soln:-

WKT Cauchy's integral formula is

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

Given:- $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$

Here $f(z) = \sin^6 z$

$a = \pi/6 = \frac{3.14}{6} = 0.523$ is lies inside $|z| = 1$

Hence by Cauchy's integral formula.

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$$= \pi i \, f''(\pi/6) \quad \text{——} (1)$$

$f(z) = \sin^6 z$

$f'(z) = 6 \sin^5 z \cos z$

$f''(z) = 6[\sin^5 z(-\sin z) + \cos z \; 5\sin^4 z \cos z]$

$f''(z) = 6[-\sin^6 z + 5 \sin^4 z \cos^2 z]$

$\sin 30° = 1/2$

$\cos 30° = \sqrt{3}/2$

$$f''(\pi/6) = 6\left[-\sin^6 \pi/6 + 5 \sin^4 \pi/6 \cos^2 \pi/6\right]$$

$$= 6\left[-(1/2)^6 + 5(1/2)^4 (\sqrt{3}/2)^2\right]$$

$$= 6\left[\frac{-1}{2^6} + \frac{5}{2^4} * \frac{3}{2^2}\right] = 6\left[\frac{-1}{2^6} + \frac{15}{2^6}\right]$$

$$\cancel{= 6\left[\frac{-1 + 5(4) + 3(16)}{2^6}\right]} = \frac{6}{2^6}\left[-1+15\right]$$

$$\cancel{= \frac{6}{2^6}\left[-1 + 20 + 48\right]} = \frac{6}{2^6}[14]$$

$$= \frac{2 \times 3 \times 2 \times 7}{2^6} = \frac{21}{2^4} = \frac{21}{16}$$

$$\therefore (1) \Rightarrow \int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \pi i \left[\frac{21}{16}\right]$$

$$= \frac{21}{16} \pi i$$

(8) Evaluate $\int_C \frac{ze^z}{(z-a)^3} dz$ where $z = a$ lies inside the closed curve c using Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a) \quad \text{——} (1)$$

Given $\int \frac{ze^z}{(z-a)^3}$

$f(z) = ze^z$

the point a lies inside c Hence by (1) we get

$$\int_C \frac{ze^z}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

$$= \pi i\, f''(a) \quad \text{—— (2)}$$

$\because f(z) = ze^z$

$f'(z) = ze^z + e^z = (z+1)e^z$

$f''(z) = (z+1)e^z + e^z = e^z(z+2)$

$\boxed{f''(a) = e^a(a+2)}$

$$\therefore (2) \Rightarrow \int_C \frac{ze^z}{(z-a)^3}\,dz = \pi i\, e^a(a+2)$$

(9) Evaluate $\int \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$ where c is $|z| = 3$

Here $f(z) = \cos \pi z^2$

$a_1 = 1$ is lies inside $|z| = 3$

$a_2 = 2$ is lies inside $|z| = 3$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z=1$ $\quad 1 = A(-1) + B(0)$

$\boxed{A = -1}$

put $z=2$ $\quad 1 = A(0) + B(2-1)$

$\boxed{B = 1}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz = -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i(-1) + 2\pi i(1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

$\because f(z) = \cos \pi z^2$

$f(1) = \cos \pi (1)^2$

$= \cos \pi = -1$

$f(2) = \cos \pi (2)^2$

$= \cos 4\pi$

$= 1$

(10) Using Cauchy's integral formula $\int_C \frac{\cos\pi z^2}{(z-1)(z-2)}\,dz$ where C is $|z| = 3/2$

Here $f(z) = \cos\pi z^2$

$a_1 = 1$ lies inside $|z| = 3/2$

$a = 2$ lies outside $|z| = 3/2$

$$\int_C \frac{\frac{\cos\pi z^2}{z-2}}{z-1}\,dz$$

Hence Here $f(z) = \frac{\cos\pi z^2}{z-2}$

$a = 1$

$$\int_C \frac{\frac{\cos\pi z^2}{z-2}}{z-1}\,dz = 2\pi i\, f(1)$$

$$= 2\pi i\,(1)$$

$$= 2\pi i$$

$\therefore f(z) = \frac{\cos\pi z^2}{z-2}$

$f(1) = \frac{\cos\pi(1)}{1-2} = \frac{\cos\pi}{-1}$

$= \frac{-1}{-1} = 1$

(11) Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$ where C is $|z| = 3$

Soln:- WKT Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \text{---(1)}$$

Given $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$

Here $f(z) = \sin\pi z^2 + \cos\pi z^2$

$a_1 = 1$ lies inside $|z| = 3$

$a_2 = 2$ lies inside $|z| = 3$

Hence by formula.

$$\int_C \frac{\frac{\sin\pi z^2 + \cos\pi z^2}{z-1}}{z-2}\,dz + \int_C \frac{\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}}{z-1}\,dz$$

$= 2\pi i\, f(2) + 2\pi i\, f(1)$

$= 2\pi i\,(1) + 2\pi i\,(1) = 2\pi i + 2\pi i = 4\pi i$ //

$f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}$

$f(2) = \frac{\sin 4\pi + \cos 4\pi}{1}$

$= \frac{1}{1} = 1$

$f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}$

$f(1) = \frac{\sin\pi + \cos\pi}{-1} = \frac{-1}{-1}$

$= 1$

(12) using Cauchy's integral formula evaluate $\int_C \frac{1}{z^2-1} dz$ where c is the circle with Centre at $z=0$ and radius 2.

Soln:

WKT Cauchy's integral formula is

$$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i \, f(a) \quad \text{—— (1)}$$

Given $\int_C \frac{1}{z^2-1} dz$ Here $f(z) = 1$

$z^2 - 1 = 0$

$z^2 = 1$

$z = \pm 1$

$$\int_C \frac{1}{(z-1)(z+1)} dz$$

$a_1 = 1$ lies on $|z| = 2$

$a_2 = -1$ lies on $|z| = 2$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$1 = A(z+1) + B(z-1)$$

$z = 1$ we get

$1 = 2A + 0$

$A = 1/2$

$z = -1$ we get

$1 = 0 - 2B$

$B = -1/2$

$$\frac{1}{z^2-1} = \frac{1/2}{z-1} + \frac{-1/2}{z+1}$$

$$\int_C \frac{1}{z^2-1} dz = 1/2 \int_C \frac{1}{z-1} dz - 1/2 \int \frac{1}{z+1} dz$$

$$= 1/2 \; 2\pi i \, f(1) - 1/2 \; 2\pi i \, f(-1)$$

$$= \pi i \, f(1) - \pi i \, f(-1)$$

$[\because f(z) = 1$

$f(1) = 1$

$f(-1) = 1]$

$$= \pi i [1 - 1]$$

$$= 0$$

(13) Evaluate $\int_C \frac{7z-1}{z^2-3z-4} dz$ where c is ellipse $x^2+4y^2=4$

$$\int_C \frac{7z-1}{z^2-3z-4} dz = \int_C \frac{7z-1}{(z-4)(z+1)} dz = \int_C \frac{\left(\frac{7z-1}{z-4}\right)}{z+1} dz$$

$[\because a_1 = 4$ lies outside $c]$

$[a_2 = -1$ lies inside $c]$

$$f(z) = \frac{7z-1}{z-4}$$

$$f(-1) = \frac{-7-1}{-1-4} = \frac{-8}{-5} = 8/5$$

$$\int_C \frac{\left(\frac{7z-1}{z-4}\right)}{z+1} dz = 2\pi i\, f(-1)$$

$$= 2\pi i \left(8/5\right)$$

$$= \frac{16}{5}\pi i$$

(14) Evaluate $\int_C \frac{dz}{z^2-7z+12}$ where c is the circle $|z| = 3.5$

$$z^2 - 7z + 12 = 0$$

$$(z-4)(z-3) = 0$$

$a_1 = 4$ lies outside $|z| = 3.5$

$a_2 = 3$ lies inside $|z| = 3.5$

$$\int_C \frac{dz}{(z-4)(z-3)} = \int_C \frac{\left(\frac{dz}{z-4}\right)}{z-3} = \int_C \frac{\left(\frac{1}{z-4}\right)}{z-3} dz$$

Here $f(z) = \frac{1}{z-4}$

$$f(3) = \frac{1}{3-4} = -1$$

$$\int_C \frac{\left(\frac{1}{z-4}\right)}{z-3} dz = 2\pi i\, f(3)$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

(15) Evaluate $\int_C \frac{z^2+1}{z^2-1} dz$ where c is $|z-1| = 1$ using Cauchy's integral formula.

Given : $|z-1| = 1$ is a circle whose centre is 1 and radius 1

$$\int_C \frac{z^2+1}{(z-1)(z+1)} dz \qquad \int_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1} dz$$

$a = 1$ lies inside c

$a = -1$ lies outside c

Here $f(z) = \frac{z^2+1}{z+1}$ is analytic inside c

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a).$$

$$\int_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz = 2\pi i\, f(1) \qquad f(z) = \frac{z^2+1}{z+1}$$

$$= 2\pi i\,(1) \qquad f(1) = \frac{1+1}{1+1} = 1$$

$$= 2\pi i$$

(16) Using Cauchy's integral formula evaluate $\int_C \frac{z}{z-2}\,dz$ where C is the circle $|z-2| = 3/2$

Soln:

Given $|z-2| = 3/2$ is a circle whose centre is 2 and radius $3/2$

$a = 2$ lies inside $|z-2| = 3/2$

Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \qquad f(z) = z$$

$$\int_C \frac{z}{z-2}\,dz = 2\pi i\, f(2) \qquad f(2) = 2$$

$$= 2\pi i\,(2)$$

$$= 4\pi i$$

(17) Using Cauchy's integral formula evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1-i| = 2$

Soln:

Given $|z+1-i| = 2$

$|z-(-1+i)| = 2$ is a circle whose centre is $-1+i$ and radius is 2

centre is $(-1, 1)$ and radius is 2.

Note: $z^2+2z+5=0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$z = -1+2i \qquad z = -1-2i$

$z^2+2z+5 = [z-(-1+2i)][z-(-1-2i)]$

$-1+2i$ (ie) $(-1, 2)$ lies inside C

$-1-2i$ (ie) $(-1, -2)$ lies outside C

$\therefore \int_c \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]} dz = \int_c \frac{\left[\frac{z+4}{z-(-1-2i)}\right]}{z-(-1+2i)} dz$ (51)

Here $f(z) = \frac{z+4}{z-(-1-2i)}$ is analytic inside c

Hence by cauchy's integral formula

$$\int_c \frac{z+4}{z^2+2z+5} dz = 2\pi i f(-1+2i)$$

$$= 2\pi i \left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right] = 2\pi i \left[\frac{3+2i}{4i}\right]$$

$$= \pi/2 \left[3+2i\right]$$

(18) Evaluate $\int_c \frac{z+4}{z^2+2z+5} dz$ where c is $|z+1+i| = 2$

Ans :- $-\pi/2 (3-2i)$

(19) Evaluate $\int_c z^2 e^{1/z} dz$ where c is $|z|=1$

Given $\int_c z^2 e^{-1/z} dz = \int_c \frac{z^2}{e^{-1/z}} dz$

$z=0$ we get $e^{-1/0} = e^{-\alpha} = 0$

0 lies inside $|z|=1$ $\quad \int_c \frac{z^2}{e^{-1/z}} dz = 2\pi i f(0) = 0$

Hence by cauchy's integral formula

(20) Evaluate $\int_c \tan z \, dz$ where c is $|z|=2$

Given $\int_c \tan z \, dz = \int_c \frac{\sin z}{\cos z} dz$

Here $f(z) = \sin z$

$\cos z = 0$

$z = \pi/2, 3\pi/2, \dots = \frac{3.14}{2}, \frac{3(3.14)}{2} = 1.57, 4.71, \dots$

1.57 lies inside $|z|=2$

other points lies out side $|z|=2$

Hence by cauchy's integral formula

$$\int_c \frac{\sin z}{\cos z} dz = 2\pi i f(\pi/2) = 2\pi i (1) = 2\pi i$$

4.7. Taylor and Laurent Expansions

Taylor Series

A function $f(z)$, analytic inside a circle c with centre at a, can be expanded in the series

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \text{ to } \infty$$

which is convergent at every point inside c.

Problems:-

① Expand $\log(1+z)$ as a Taylor's series about $z=0$

Let $f(z) = \log(1+z)$

$f'(z) = \frac{1}{1+z}$ $\qquad f(0) = \log 1 = 0$

$f''(z) = \frac{-1}{(1+z)^2}$ $\qquad f'(0) = \frac{1}{1+0} = 1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad f''(0) = \frac{-1}{1} = -1$

$f^{IV}(z) = \frac{-6}{(1+z)^4}$ $\qquad f'''(0) = 2$

$f^{IV}(0) = -6$

So the Taylor's series for $\log(1+z)$ about $z=0$ is

$$\log(1+z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$= 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad |z|<1$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

② Find the Taylor's series of $f(z) = \tanh z$ about the point $z=0$

Soln: Taylor's series of $f(z)$ about $z = z_0$ is given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$f(z) = \tanh z$ $\qquad f(0) = 0$

$f'(z) = \operatorname{sech}^2 z$ $\qquad f'(0) = 1$

$f''(z) = 2\,\text{sech}\,z\,(-\text{sech}\,z \tanh z)$ $\qquad f''(0) = 0$

$f''(z) = -2\,\text{sech}^2 z \tanh z$ $\qquad f'''(0) = -2$

$f'''(z) = -2\left[\text{sech}^2 z(\text{sech}^2 z) + \tanh z \; 2\,\text{sech}\,z(-\text{sech}\,z \tanh z)\right]$

$= -2\left[\text{sech}^4 z - 2\,\text{sech}^2 z \tanh^2 z\right]$

$\ldots\ldots$

$\therefore f(z) = 0 + z(1) + \frac{z^2}{2!}(0) + \frac{z^3}{3!}(-2) + \cdots$

$= z - \frac{z^3}{3} + \cdots \quad |z| < \infty$

③ Find the Taylor's expansion about $z=0$ of $f(z) = \frac{z}{(z+1)(z-3)}$

Splitting $f(z)$ into partial fractions

$$\frac{z}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z+1)$$

put $z=-1$ we get

$-1 = -4A + B(0)$

$\boxed{A = \frac{1}{4}}$

put $z=3$ we get

$3 = 4B$

$\boxed{B = \frac{3}{4}}$

$\therefore f(z) = \frac{1/4}{z+1} + \frac{3/4}{z-3}$

$= \frac{1}{4(z+1)} + \frac{3}{4(z-3)}$

$= \frac{1}{4}(1+z)^{-1} + \frac{3}{4}\,\frac{1}{-3\left[1 - \frac{z}{3}\right]}$

$= \frac{1}{4}(1+z)^{-1} - \frac{1}{4}\left[1 - \frac{z}{3}\right]^{-1}$

$\therefore f(z) = \frac{1}{4}\left[(1+z)^{-1} - \left(1 - \frac{z}{3}\right)^{-1}\right]$

$= \frac{1}{4}\left[\left[1 - z + z^2 - z^3 + z^4 \ldots\right] - \left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right]\right]$

$= \frac{1}{4}\left[\left((-1)^1 - \frac{1}{3}\right)z + \left[(-1)^2 - \frac{1}{3^2}\right]z^2 + \left[(-1)^3 - \frac{1}{3^3}\right]z^3 + \cdots\right]$

$= \frac{1}{4}\sum_{n=1}^{\infty}\left[(-1)^n - \frac{1}{3^n}\right]z^n$

④ Expand $f(z) = \sin z$ using Taylor's series at $z = \pi/4$

Soln:

$f(z) = \sin z$ $\quad f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$ $\qquad$ Here $a = \pi/4$

$f'(z) = \cos z$ $\quad f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$f''(z) = -\sin z$ $\quad f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$

$f'''(z) = -\cos z$ $\quad f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$

. . . .

Taylor's series is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

$$= 1/\sqrt{2} + \frac{1/\sqrt{2}}{1!}(z-\pi/4) - \frac{1/\sqrt{2}}{2!}(z-\pi/4)^2 + \frac{-1/\sqrt{2}}{3!}(z-\pi/4)^3 + \cdots$$

$$= 1/\sqrt{2} + 1/\sqrt{2}\,(z-\pi/4) - \frac{1}{2\sqrt{2}}(z-\pi/4)^2 - \frac{1}{6\sqrt{2}}(z-\pi/4)^3 + \cdots$$

⑤ Expand $f(z) = e^z$ as Taylor's series at $z = 0$

$f(z) = e^z$ $\quad f(0) = 1$

$f'(z) = e^z$ $\quad f'(0) = 1$

$f''(z) = e^z$ $\quad f''(0) = 1$

$f'''(z) = e^z \ldots$ $\quad f'''(0) = 1 \ldots$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

$$= 1 + \frac{1}{1!}(z-0) + \frac{1}{2}(z-0)^2 + \frac{1}{6}(z-0)^3 + \cdots$$

$$= 1 + z + z^2/2 + z^3/6 + \cdots$$

⑥ Find the Taylor's series for $1/z$ at the point $z = 1$

$f(z) = 1/z = z^{-1}$ $\qquad f(1) = 1$

$f'(z) = (-1)\, z^{-2} = -1/z^2$ $\qquad f'(1) = -1/1 = -1$

$f''(z) = (-1)(-2)\, z^{-3} = 2/z^3$ $\qquad f''(1) = 2$

$f'''(z) = (-1)(-2)(-3)\, z^{-4} = -6/z^4$ $\qquad f'''(1) = -6$

$f^{iv}(z) = (-1)(-2)(-3)(-4)\, z^{-5} = \frac{24}{z^5}$ $\qquad f^{iv}(1) = 24$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

$$= 1 + \frac{-1}{1!}(z-1) + \frac{2}{2!}(z-1)^2 + \frac{-6}{3!}(z-1)^3 + \frac{24}{4!}(z-1)^4 + \cdots$$

$$= 1 - (z-1)^1 + (z-1)^2 - (z-1)^3 + (z-1)^4 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Laurent's series

Let c_1, c_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let $f(z)$ be analytic on c_1 and c_2 and in the annular region R between them. Then, for any point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{n+1}}\,dz$

$b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{1-n}}\,dz$

Problems

① Obtain the Laurent's series for $f(z) = \frac{1}{z(z-1)}$ for (i) $0 < |z| < 1$

(ii) $0 < |z-1| < 1$

Soln:

Let $\frac{1}{z(z-1)} = A/z + \frac{B}{z-1}$

$1 = A(z-1) + B(z)$

put $z = 0$ $z = 1$

$1 = -A$ $1 = B$

$\boxed{A = -1}$ $\boxed{B = 1}$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$= \frac{-1}{z} + \frac{1}{z-1}$$

(i) $|z|<1$

If the denominator contains the same term as in the condition then there is no series for such term.

$\therefore$ There is no series for $-\frac{1}{z}$

we can write $\frac{1}{z(z-1)}$ as $\frac{-1}{z}+\frac{1}{-1+z} = \frac{-1}{z}+\frac{1}{-(1-z)}$

$$= -\frac{1}{z}-(1-z)^{-1}$$

$$= -\frac{1}{z}-(1+z+z^2+\dots)$$

(ii) $|z-1|<1$

put $u = z-1$

$z = u+1$

$|u|<1$

$$\frac{1}{z(z-1)} = \frac{-1}{z}+\frac{1}{z-1}$$

$$= \frac{-1}{u+1}+\frac{1}{u+1-1}$$

$$= -1(1+u)^{-1}+\frac{1}{u}$$

$$= -1[1-u+u^2-\dots]+\frac{1}{u}$$

$$= -1[1-(z-1)+(z-1)^2+\dots]+\frac{1}{z-1}$$

$=$

② Find the Laurent's series expansion of $f(z)=\frac{1}{z^2+3z+2}$ in the region $1<|z|<2$

Soln:- Given $\frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)}$

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1}+\frac{B}{z+2}$$

$$1 = A(z+2)+B(z+1)$$

put $z=-1$ we get

$1 = A+B(0)$

$\boxed{A=1}$

put $z=-2$ we get

$1 = A(0)+B(-1)$

$\boxed{B=-1}$

$$\therefore f(z) = \frac{1}{z+1}-\frac{1}{z+2}$$

Given $1 < |z| < 2$

$1 < |z|$ $\qquad$ $|z| < 2$

$\left|\frac{1}{z}\right| < 1$ $\qquad$ $\left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = \frac{1}{z(1+1/z)} - \frac{1}{2(1+z/2)}$$

$$= \frac{1}{z}(1+1/z)^{-1} - \frac{1}{2}(1+z/2)^{-1}$$

$$= 1/z\left[1 - 1/z + (1/z)^2 - (1/z)^3 + \cdots\right] - 1/2\left[1 - (z/2) + (z/2)^2 - (z/2)^3 + \cdots\right]$$

(9) Expand $f(z) = \frac{z}{(z-1)(z-3)}$ as Laurent's Series valid in the following regions

(i) $1 < |z| < 3$ (ii) $0 < |z-1| < 2$

Soln:

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z-1)$$

put $z = 1$ we get $\qquad$ put $z = 3$ we get

$1 = A(1-3) + B(0)$ $\qquad$ $3 = A(0) + B(3-1)$

$1 = -2A$ $\qquad$ $3 = 2B$

$\boxed{A = -1/2}$ $\qquad$ $\boxed{B = 3/2}$

$$\frac{z}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

(i) $1 < |z| < 3$

$1 < |z|$; $|z| < 3$

$\left|\frac{1}{z}\right| < 1$ $\qquad$ $\left|\frac{z}{3}\right| < 1$

$$f(z) = \frac{1}{2}\left(\frac{1}{1-z}\right) + \frac{3}{2}\left(\frac{1}{z-3}\right)$$

$$= \frac{1}{2}\,\frac{1}{(-z)(1-1/z)} + \frac{3}{2}\,\frac{1}{(-3)\left(1-z/3\right)}$$

$$= \frac{-1}{2z}\left(1-1/z\right)^{-1} - \frac{1}{2}\left(1-z/3\right)^{-1}$$

$$= \frac{-1}{2z}\left[1+1/z+(1/z)^2+(1/z)^3+\cdots\right] - \frac{1}{2}\left[1+z/3+(z/3)^2+(z/3)^3+\cdots\right]$$

(ii) $0<|z-1|<2$

put $u = z-1$

$z = u+1$

$0<|u|<2$ (or) $|u/2|<1$

$$f(z) = \frac{1}{2}\left(\frac{1}{1-z}\right) + \frac{3}{2}\left(\frac{1}{z-3}\right) = \frac{1}{2}\left(\frac{1}{1-u-\cancel{1}}\right) + \frac{3}{2}\left(\frac{1}{u+1-3}\right)$$

$$= \frac{-1}{2u} + \frac{3}{2}\left(\frac{1}{u-2}\right)$$

$$= \frac{-1}{2u} + \frac{3}{2}\,\frac{1}{(-2)\left(1-u/2\right)}$$

$$= \frac{-1}{2u} - \frac{3}{4}\left(1-u/2\right)^{-1} \Rightarrow \frac{-1}{2u} - \frac{3}{4}\left[1-(u/2)+(u/2)^2+\cdots\right]$$

$$= \frac{-1}{2u} - \frac{3}{4}\left[1+\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2+\cdots\right]$$

$$= \frac{-1}{2(z-1)} - \frac{3}{4}\left[1+\left(\frac{z-1}{2}\right)+\frac{(z-1)^2}{2}+\cdots\right]$$

④ Find the Laurant's series which represents the function

$\dfrac{z}{(z+1)(z+2)}$ in (i) $|z|>2$ (ii) $|z+1|<1$

Soln:

let $f(z) = \dfrac{z}{(z+1)(z+2)}$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

put $z=-1$ we get	put $z=-2$ we get
$-1 = A(-1+2)+B(0)$	$-2 = A(0)+B(-2+1)$
$-1 = A$ $\boxed{A=-1}$	$-2 = -B$
	$\boxed{B=2}$

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2}$$

(i) $|z| > 2$ ie $2 < |z|$; $\frac{2}{|z|} < 1 \Rightarrow \frac{1}{|z|} < 1$

$$f(z) = \frac{-1}{1+z} + \frac{2}{z(1+\cancel{2}/z)}$$

$$= \frac{-1}{z(1+1/z)} + \frac{2}{\cancel{z}(1+2/z)}$$

$$= -\frac{1}{z}(1+1/z)^{-1} + \frac{2}{z}(1+2/z)^{-1}$$

$$= \frac{1}{z}\left[-1 + \frac{1}{z} - (1/z)^2 + \cdots\right] + \frac{2}{z}\left[1 - 2/z + (2/z)^2 + \cdots\right]$$

(ii) $|z+1| < 1$ (ie) $|u| < 1$

Let $u = z+1$

$u+1 = z+2$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{u} + \frac{2}{1+u}$$

$$= -1/u + 2(1+u)^{-1} = -1/u + 2\left[1 - u + u^2 - u^3 + \cdots\right]$$

$$= \frac{-1}{1+z} + 2\left[1 - (1+z) + (1+z)^2 - (1+z)^3 + \cdots\right]$$

4.8 SINGULARITIES - CLASSIFICATION - RESIDUES.

Singularities - classification

(1) Zero of an analytic function

If a function $f(z)$, analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

(2) Simple zero

If $f(z_0) = 0$ and $f'(z_0) \neq 0$ then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order

(3) zero of order n

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^{n}(z_0) \neq 0$ then z_0 is called zero of order n.

problems:

① Find the zeros of $f(z) = \dfrac{z^2+1}{1-z^2}$

soln The zeros of $f(z)$ are given by $f(z) = 0$

$$f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$$

(ie) $(z+i)(z-i) = 0$

$z = i$ is a simple zero

$z = -i$ is a simple zero

② Find the zeros of $\dfrac{z^3-1}{z^3+1}$

soln: The zeros of $f(z)$ are given by $f(z) = 0$

$$\frac{z^3-1}{z^3+1} = 0 \qquad z^3 - 1 = 0$$

$z = (1)^{1/3} = 1, w, w^2$ [cube roots of unity]

(3) Find the zeros of $\frac{\sin z - z}{z^3}$

Let $f(z) = \frac{\sin z - z}{z^3}$

zeros of $f(z)$ are given by $f(z) = 0$

$$f(z) = \frac{\sin z - z}{z^3} = \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] - z}{z^3}$$

$$= \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^3} = \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

Now

$$\lim_{z \to 0} \frac{\sin z - z}{z^3} \neq 0$$

$\therefore f(z)$ has no zeros

Removable Singularity

If the principal part of $f(z)$ contains no term (i.e.) $b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Poles

If we can find a positive integer n such that $\lim_{z \to a} (z-a)^n f(z) \neq 0$ then $z = 0$ is called a pole of order n for $f(z)$

Essential Singularity

If the principal part contains an infinite number of non zero terms, then $z = z_0$ is known as a essential singularity.

Evaluation of Residues

Residue at a pole of order m

If $z = z_0$ is a pole of order m, a simple formula to determine the residue given by

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

If $z = z_0$ is a simple pole of $f(z)$ and if $f(z) = \dfrac{\phi(z)}{\psi(z)}$ then $Res[f(z), z_0] = \dfrac{\phi(z_0)}{\psi'(z_0)}$

if $z = z_0$ is a simple pole of $f(z)$ then $Res[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

problems

① classify the singularity of the function

$$f(z) = \frac{z-2}{z^2} \sin\left(\frac{1}{z-1}\right)$$

Soln:

poles of $f(z)$ are obtained by equating the denominator to zero

$z^2 = 0$

$z = 0$ is a pole of order 2.

zeros of $f(z)$ are obtained by equating the numerator to zero.

$$(z-2) \sin\left(\frac{1}{z-1}\right) = 0$$

$z - 2 = 0$ and $\sin\left(\frac{1}{z-1}\right) = 0$

$z = 2$ is a simple zero

$\sin\left(\frac{1}{z-1}\right) = 0 \qquad \frac{1}{z-1} = n\pi$

$z - 1 = \frac{1}{n\pi}$ (ie) $z = \frac{1}{n\pi} + 1 \qquad n = 0, \pm 1, \pm 2, \ldots$

* The limit of zero given by 1

$\therefore z = 1$ is isolated an essential singularity.

② classify the singularity of $f(z) = \dfrac{e^{1/z}}{(z-a)^2}$

Soln:

poles of $f(z)$ are obtained by equating the denominator to zero

$(z-a)^2 = 0$

$z = a$ is a pole of order 2.

Now zeros of $f(z)$

$$\lim_{z \to 0} \frac{e^{1/z}}{(z-a)^2} = \frac{\alpha}{a^2} = \alpha \neq 0$$

$\Rightarrow z = 0$ is a removable singularity

$\therefore f(z)$ has no zeros.

③ classify the nature of singularities of the functions $\frac{e^z}{z^2+4}$ and $e^{1/z}$

Soln:

Poles of $f(z)$ are obtained by equating the denominator to zero

$$z^2+4 = 0$$

$$z^2 = -4$$

$$z = \pm 2i$$

$z = 2i$ is a simple pole and

$z = -2i$ is another simple pole

$$\underset{z \to 0}{Lt} \frac{e^z}{z^2+4} = 1/4 \neq 0$$

$\therefore z=0$ is a removable singularity

$\therefore f(z)$ has no zeros.

(ii) Let $f(z) = e^{1/z}$

$z=0$ as an essential singularity since $f(z)$ is an infinite series of negative power of z.

$$e^{1/z} = 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \cdots$$

④ classify the various types of singularities and give one example for each type.

Soln:

Isolated singularity

A point $z=z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z=z_0$

(ii) There exists a neighbourhood of $z=z_0$ containing no other singularity

Example

$f(z) = 1/z$

This function is analytic everywhere except at $z=0$

$\therefore z=0$ is an isolated singularity.

Removable singularity

A singular point $z=z_0$ is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists finitely

$$f(z) = \frac{\sin z}{z}$$

Poles:-

If we can find a positive integer n such that

$\lim_{z \to z_0} (z-z_0)^n f(z) \neq 0$ then $z = z_0$ is called a pole of order n for $f(z)$.

Example:-

$$f(z) = \frac{1}{(z-4)^2 (z-3)^4}$$

Here $z = 4$ is a pole of order 2

$z = 3$ is a pole of order 4.

Essential singularity

If the principal part contains an infinite number of terms, then $z = z_0$ is known as a essential singularity

Example: $f(z) = e^{1/z}$.

Problems:-

① Calculate the residue of $f(z) = \frac{1-e^{2z}}{z^3}$

Given $f(z) = \frac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3

$$b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \quad \text{Here } m=3$$

$$\text{Res}(z=0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[(z-0)^3 \frac{1-e^{2z}}{z^3} \right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[1 - e^{2z} \right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz} \left[-2e^{2z} \right] = \frac{1}{2!} \lim_{z \to 0} \left[-4e^{2z} \right] = \frac{1}{2}(-4)$$

$$= -2.$$

② Test for singularity of $\frac{1}{z^2+1}$ and hence find the corresponding residues

Soln:-

Let $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

Here $z = -i$ is a simple pole

$z = i$ is a simple pole.

$\text{Res}[z=i] = \underset{z\to i}{lt}\ (z-i)\, f(z) = \underset{z\to i}{lt}\ (z-i)\dfrac{1}{(z+i)(z-i)} = \underset{z\to i}{lt}\ \dfrac{1}{z+i}$

$= \dfrac{1}{i+i} = \frac{1}{2i}$

$\text{Res}[z=-i] = \underset{z\to -i}{lt}\ (z+i)\, f(z) = \underset{z\to -i}{lt}\ (z+i)\dfrac{1}{(z+i)(z-i)} = \underset{z\to -i}{lt}\ \dfrac{1}{-i-i} = \dfrac{-1}{2i}$

(3) Find the residue of $f(z) = \dfrac{z}{(z-1)^2}$ at its pole

Soln:

Given $f(z) = \dfrac{z}{(z-1)^2}$

$z = 1$ is a pole order 2.

$\text{Re}[z=z_0] = \underset{z\to z_0}{lt}\ \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$ Here $m=2$

$\text{Res}[z=1] = \underset{z\to 1}{lt}\ \dfrac{1}{1!}\dfrac{d}{dz}\left[(z-1)^2 \cdot \dfrac{z}{(z-1)^2}\right]$

$= \underset{z\to 1}{lt}\ \frac{d}{dz}[z] = \underset{z\to 1}{lt}\ (1) = 1$

(4) Calculate the residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ at its pole.

Soln:

Given $f(z) = \dfrac{e^{2z}}{(z+1)^2}$

Here $z=-1$ is a pole of order 2

W.K.T. $\text{Res}(z=z_0) = \underset{z\to z_0}{lt}\ \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$

$\text{Res}[z=-1] = \underset{z\to -1}{lt}\ \dfrac{1}{1!}\dfrac{d}{dz}\left[(z+1)^2 \dfrac{e^{2z}}{(z+1)^2}\right]$

$= \underset{z\to -1}{lt}\ \frac{d}{dz}\left[e^{2z}\right] = \underset{z\to -1}{lt}\ 2e^{2z} = 2e^{-2}$

Repeated Sum

(5) Test for singularity of $\dfrac{1}{z^2+1}$ and hence find the corresponding residues

Soln:

Let $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$

Here $z=-i$ is a simple pole

$z = i$ is a simple pole

$\text{Res}[z=i] = \underset{z\to i}{lt}\ (z-i)\, f(z) = \underset{z\to i}{lt}\ (z-i)\dfrac{1}{(z-i)(z+i)} = \underset{z\to i}{lt}\ \dfrac{1}{z+i} = \dfrac{1}{2i}$

$$Res[z=-i] = \lim_{z\to -i} (z+i) f(z) = \lim_{z\to -i} (z+i) \frac{1}{(z+i)(z-i)} = \lim_{z\to -i} \frac{1}{(z-i)}$$

$$= \frac{-1}{2i}$$

(5) Find the residue of $\frac{z^3}{(z-1)^4 (z-2)(z-3)}$

Let $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

Here
$z=1$ is a pole of order 4
$z=2$ is a pole of order 1
$z=3$ is a pole of order 1

(i) $$Res[z=2] = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} (z-2) \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \lim_{z\to 2} \frac{z^3}{(z-1)^4 (z-3)}$$

$$= \frac{8}{(1)^4(-1)} = -8$$

(ii) $$Res[z=3] = \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} (z-3) \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \lim_{z\to 3} \frac{z^3}{(z-1)^4 (z-2)}$$

$$= \frac{27}{(2^4)(1)} = \frac{27}{16}$$

(iii) $$Res(z=1) = \lim_{z\to 1} \frac{1}{3!} \frac{d^3}{dz^3} \left[(z-1)^4 f(z) \right]$$

$$= \lim_{z\to 1} \frac{1}{6} \frac{d^3}{dz^3} \left[(z-1)^4 \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right]$$

$$= \lim_{z\to 1} \frac{1}{6} \frac{d^3}{dz^3} \left[\frac{z^3}{(z-2)(z-3)} \right]$$

$$\begin{array}{r|l} & z+5 \\ \hline z^2-5z+6 & z^3 \\ & \underset{(-)}{z^3} \overset{(+)}{-5z^2} \overset{(-)}{+6z} \\ \hline & 5z^2 - 6z \\ & \underset{(-)}{5z^2} \overset{(+)}{-25z} \overset{(-)}{+30} \\ \hline & 19z - 30 \end{array}$$

$$\frac{z^3}{(z-2)(z-3)} = z + 5 + \frac{19z-30}{(z-2)(z-3)}$$

$$\frac{19z-30}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$19z - 30 = A(z-3) + B(z-2)$$

Put $z=3$
$57 - 30 = B$
$\boxed{B = 27}$

Put $z=2$
$38 - 30 = -A$
$8 = -A$
$\boxed{A = -8}$

$$\frac{z^3}{(z-2)(z-3)} = z+5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\text{Res}(z=1) = \lim_{z\to 1} \frac{1}{6} \frac{d^3}{dz^3}\left[(z+5) - \frac{8}{z-2} + \frac{27}{(z-3)}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d^2}{dz^2}\left[1 + \frac{8}{(z-2)^2} - \frac{27}{(z-3)^2}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d}{dz}\left[\frac{-16}{(z-2)^3} + \frac{54}{(z-3)^3}\right]$$

$$= \frac{1}{6} \lim_{z\to 1}\left[\frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}\right]$$

$$= \frac{1}{6}\left[\frac{48}{(-1)^4} - \frac{162}{(-2)^4}\right]$$

$$= \frac{1}{6}\left[48 - \frac{162}{16}\right] = \frac{1}{6}\left[\frac{768-162}{16}\right] = \frac{1}{6}\left[\frac{606}{16}\right] = \frac{303}{3\times 16}$$

$$= \frac{101}{16}$$

(6) Find the residue of $\cot z$ at $z=0$

Let $f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$.

Here $z=0$ is a simple pole

$\phi(z) = \cos z$ $\qquad \psi(z) = \sin z$

$\phi(0) = \cos 0 = 1$ $\qquad \psi'(z) = \cos z$

$\psi'(0) = 1$

$$\therefore \text{Res}[f(z), 0] = \frac{\phi(0)}{\psi'(0)} = \frac{1}{1} = 1$$

(7) Find the residues of $f(z) = \frac{1}{(z^2+a^2)^2}$ at its singularities

$$f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ia)^2(z-ia)^2}$$

Here $z = ia$ is a pole of order 2

$z = -ia$ is a pole of order 2

$$\text{Res}[f(z), ia] = \lim_{z\to ia} \frac{1}{1!} \frac{d}{dz}\left[(z-ia)^2 f(z)\right]$$

$$= \lim_{z\to ia} \frac{d}{dz}\left[(z-ia)^2 \frac{1}{(z+ia)^2(z-ia)^2}\right] = \lim_{z\to ia} \frac{d}{dz}\left[\frac{1}{(z+ia)^2}\right]$$

$$= \lim_{z \to ia} \left[\frac{0 - 2(z+ia)}{(z+ia)^4} \right] = \lim_{z \to ia} \left[\frac{-2}{(z+ia)^3} \right] = \frac{-2}{(ai+ia)^3} = \frac{-2}{(2ia)^3}$$

$$\text{Res}\left[f(z), -ia\right] = \lim_{z \to -ia} \frac{1}{1!} \frac{d}{dz}\left[(z+ia)^2 f(z)\right]$$

$$= \lim_{z \to -ia} \frac{d}{dz}\left[(z+ia)^2 \frac{1}{(z+ia)^2 (z-ia)^2}\right] = \lim_{z \to -ia} \frac{d}{dz}\left[\frac{1}{(z-ia)^2}\right]$$

$$= \lim_{z \to -ia} \left[\frac{-2}{(z-ia)^3} \right] = \frac{-2}{(-2ia)^3} = \frac{2}{8i^3 a^3} = \frac{1}{-4ia^3} =$$

$$\frac{-1}{4ia^3} = \frac{i^2}{4ia^3} = \frac{i}{4a^3} \,//$$

(8) Use Laurent's series to find the residue of $\frac{e^{2z}}{(z-1)^2}$ at $z=1$

$$\text{Let } f(z) = \frac{e^{2z}}{(z-1)^2} = \frac{e^{2(z-1)+2}}{(z-1)^2} = e^2 \frac{e^{2(z-1)}}{(z-1)^2}$$

$$= \frac{e^2}{(z-1)^2}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2 (z-1)^2}{2!} + \frac{2^3 (z-1)^3}{3!} + \cdots\right]$$

$$= e^2 \left[\frac{1}{(z-1)^2} + \frac{2}{(z-1)} + 2 + \frac{4}{3}(z-1) + \cdots\right]$$

This is Laurent's series expansion of $f(z)$ at $z=1$

$\text{Res}[f(z), 1]$ = coefficient of $\frac{1}{z-1}$ in Laurent's expansion

$$= 2e^2 \,//$$

Cauchy's residue theorem

State and prove Cauchy's theorem on residues

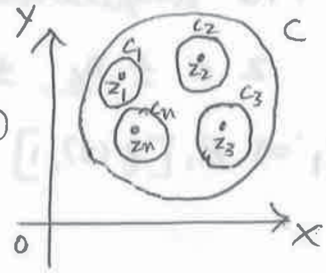
Statement:

If $f(z)$ be analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2 \ldots z_n$ inside C, then $\int_C f(z)\,dz = 2\pi i$ [Sum of the residues of $f(z)$ at $z_1, z_2, \ldots z_n$] $= 2\pi i \sum_{i=1}^{n} R_i$ [Where is the residue of $f(z)$ at $z = z_i$]

Proof:-

we enclose the singularities $z_1\ z_2 \dots z_n$ by small non-intersecting circles $c_1\ c_2 \dots c_n$ with centres at $z_1\ z_2 \dots z_n$ and radii $\rho_1\ \rho_2 \dots \rho_n$ lying wholly inside c.

Then $f(z)$ is analytic in the multiply connected region enclosed by the curves $c, c_1, c_2 \dots c_n$. Hence by Cauchy's extension of integral theorem.



$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \dots + \int_{c_n} f(z)\,dz \quad \text{— (1)}$$

Now $z = z_i$ is an isolated singularity

Hence by Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_i)^n + \sum_{n=1}^{\infty} b_n (z-z_i)^{-n}$$

where $b_n = \dfrac{1}{2\pi i} \displaystyle\int_{c_i} \dfrac{f(z)\,dz}{(z-z_i)^{1-n}}$

$\therefore$ Residue of $f(z)$ at $z = z_i$ is $b_1 = \dfrac{1}{2\pi i} \displaystyle\int_{c_i} f(z)\,dz$

$$\text{(ie)} \quad \int_{c_1} f(z)\,dz = 2\pi i b_1 = 2\pi i R_1 \quad \text{— (2)}$$

Hence using (2) in (1) we get

$$\int_c f(z)\,dz = 2\pi i \left(R_1 + R_2 + \dots + R_n\right) = 2\pi i \sum R_i$$

Problems:-

(1) Evaluate $\displaystyle\int_c \frac{z \sec z\,dz}{1-z^2}$ where c is the ellipse $4x^2 + 9y^2 = 9$

Soln

Let $f(z) = \dfrac{z \sec z}{1-z^2} = \dfrac{z}{(1-z)(1+z)\cos z}$

Singular points of the function $f(z)$ are got by equating the denominator zero

$\cos z = 0, \quad 1-z = 0, \quad 1+z = 0$

$z = (2n+1)\pi/2 \quad n = 0, \pm 1, \pm 2, \pm 3 \dots$

$z = \pm \pi/2, \pm 3\pi/2, \dots$

$z = 1$ Given $4x^2 + 9y^2 = 9$

$z = -1$

$$\frac{x^2}{(3/2)^2} + \frac{y^2}{1} = 1$$

The ellipse meets the real axis at $\pm 3/2$ and the imaginary axis at $y = \pm 1$

The singularities $z = \pm 1$ lie inside c

$z = \pm \pi/2, \pm 3\pi/2, \ldots$ lie outside c

$$\text{Res}_1 \Rightarrow R_1[f(z), 1] = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{z}{(1-z)(1+z)\cos z} = \lim_{z \to 1} \cancel{(1-z)} \frac{-z}{\cancel{(1-z)}(1+z)\cos z}$$

$$= \lim_{z \to 1} \frac{-z}{(1+z)\cos z} = \frac{-1}{2\cos 1} = -\tfrac{1}{2} \sec 1$$

$$R_2[f(z), -1] = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} \cancel{(z+1)} \frac{z}{(1-z)\cancel{(1+z)}\cos z} = \lim_{z \to -1} \frac{z}{(1-z)\cos z} = \frac{-1}{2\cos(-1)}$$

$$= \frac{-1}{2\cos 1} = -\tfrac{1}{2}\sec 1$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2] = 2\pi i\left[-\tfrac{1}{2}\sec 1 - \tfrac{1}{2}\sec 1\right]$$

$$= -2\pi i \sec 1$$

② If c is the circle $|z| = 3$ evaluate $\int_c \tan z\,dz$

Soln:

Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

singular points of the function $f(z)$ are got by equating the denominator to zero

$\cos z = 0$

$z = (2n+1)\pi/2 \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

$z = \pi/2$ lies inside c

$z = -\pi/2$ lies inside c

other points lie outside c

$$\therefore R_1\left[f(z), \tfrac{\pi}{2}\right] = \frac{\phi(\pi/2)}{\psi'(\pi/2)}$$

Here $f(z) = \dfrac{\phi(z)}{\psi(z)} = \dfrac{\sin z}{\cos z}$

$\phi(z) = \sin z$ $\qquad$ $\psi(z) = \cos z$

$\phi(\pi/2) = 1$ $\qquad$ $\psi'(z) = -\sin z$

$\phi(-\pi/2) = -1$ $\qquad$ $\psi'(\pi/2) = -1$

$\psi'(-\pi/2) = 1$

$$R_1\left[f(z), \pi/2\right] = \frac{1}{-1} = -1$$

$$R_2\left[f(z), -\pi/2\right] = \frac{\phi(-\pi/2)}{\psi'(-\pi/2)} = \frac{-1}{1} = -1$$

$\therefore$ By Cauchy's residue theorem

$$\int_c \tan z \, dz = 2\pi i \,[\text{sum of the residues}]$$

$$\int_c \tan z \, dz = 2\pi i \,[R_1 + R_2]$$

$$= 2\pi i\,[-1-1] = -4\pi i$$

③ If C is the circle $|z| = 3$ then evaluate $\displaystyle\int_c \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}\,dz$

Soln:

Let $f(z) = \dfrac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}$

Singular points of the function $f(z)$ are got by equating the denominator to zero we get

$(z+1)(z+2) = 0$

$z = -1$ lies inside $|z| = 3$

$z = -2$ lies inside $|z| = 3$

$$R_1\left[f(z), -1\right] = \lim_{z\to -1} (z+1) f(z)$$

$$= \lim_{z\to -1} (z+1) \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} = \lim_{z\to -1} \frac{\cos \pi z^2 + \sin \pi z^2}{(z+2)}$$

$$= \frac{\cos \pi + \sin \pi}{1} = \frac{-1+0}{1} = -1$$

$$R_2[f(z), -2] = \lim_{z \to -2} (z+2) f(z) = \lim_{z \to -2} (z+2) \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}$$

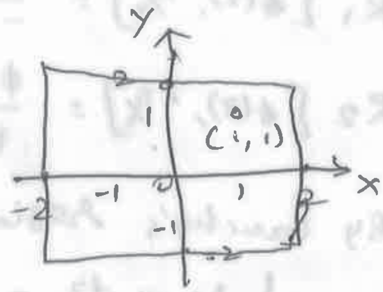
$$= \lim_{z \to -2} \frac{\cos \pi z^2 + \sin \pi z^2}{z+1} = \frac{\cos 4\pi + \sin 4\pi}{-1} = \frac{1+0}{-1} = -1$$

By Cauchy's Residue theorem

$$\int_C f(z) dz = 2\pi i \,[\text{Sum of the residues}]$$

$$= 2\pi i\,[-1-1] = -4\pi i$$

④ If C is the boundary of the square whose sides along the lines $x = \pm 2$ and $y = \pm 2$ and described in the positive sense find the value of $\int_C \frac{\tan z/2}{(z-1-i)^2} dz$



soln:

Let $f(z) = \frac{\tan z/2}{(z-1-i)^2}$

The poles are given by

$$[z-(1+i)]^2 = 0$$

$z = 1+i$ is a Pole of order 2

The pole $z = 1+i$ lies inside C

$$\text{Res}[f(z), 1+i] = \frac{1}{(2-1)!} \lim_{z \to 1+i} \frac{d}{dz} [z-(1+i)]^2 f(z)$$

$$= \lim_{z \to 1+i} \frac{d}{dz} \left[\cancel{[z-(1+i)]^2} \frac{\tan z/2}{\cancel{[z-(1+i)]^2}} \right]$$

$$= \lim_{z \to 1+i} \frac{d}{dz} \left[\tan z/2\right]$$

$$= \lim_{z \to 1+i} \left[\sec^2 z/2\right]\left[\tfrac{1}{2}\right] = \tfrac{1}{2} \sec^2 \left(\frac{1+i}{2}\right)$$

By Cauchy's Residue theorem

$$\int_C f(z) dz = 2\pi i\,[\text{Sum of the residues}]$$

$$= \cancel{2}\pi i \, \tfrac{1}{\cancel{2}} \sec^2\left(\frac{1+i}{2}\right) = \pi i \sec^2\left(\frac{1+i}{2}\right)$$

4.9 CONTOUR INTEGRATION

CIRCULAR AND SEMI-CIRCULAR CONTOURS

Type 1

Problems Based on contour integration $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

① Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by contour integration

Soln:

Let $z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta = iz\, d\theta$

$d\theta = \frac{1}{iz}\, dz$

$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right] = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

Given $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{1}{2+\frac{1}{2}\left[\frac{z^2+1}{z}\right]} \frac{dz}{iz}$ where c is $|z| = 1$

$$= \int_c \frac{2z}{4z + z^2 + 1} \frac{dz}{iz} = \frac{2}{i} \int_c \frac{dz}{z^2+4z+1}$$

Let $f(z) = \frac{1}{z^2+4z+1}$

$$= \frac{1}{[z-(-2+\sqrt{3})][z-(-2-\sqrt{3})]}$$

$z = -2+\sqrt{3} = -2+1.7321 = -0.268$ is a simple pole which lies inside c

$z = -2-\sqrt{3} = -2-1.7321 = -3.7321$ is a simple pole which lies outside c

$\text{Res}[f(z), -2+\sqrt{3}] = \lim_{z\to(-2+\sqrt{3})} [z-(-2+\sqrt{3})]\, f(z)$

$$= \lim_{z\to(-2+\sqrt{3})} \cancel{[z-(-2+\sqrt{3})]} \frac{1}{\cancel{[z-(-2+\sqrt{3})]}[z-(-2-\sqrt{3})]}$$

$$= \lim_{z\to(-2+\sqrt{3})} \frac{1}{z-(-2-\sqrt{3})} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

By cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i\,[\text{sum of the residues}]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{3}}\right] = \frac{\pi i}{\sqrt{3}}$$

Therefore $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\left[\frac{\pi i}{\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$

② using contour integration evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Soln

let $z = e^{i\theta}$

$dz = i e^{i\theta}\,d\theta = iz\,d\theta$

$d\theta = \frac{1}{iz}\,dz$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right] = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$\sin\theta = \frac{1}{2i}\left[z-\frac{1}{z}\right] = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

Given $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = R.P\int_c \frac{1}{13+5\frac{1}{2i}\left[\frac{z^2-1}{z}\right]}\frac{dz}{iz}$ where c is $|z|=1$

$$= R.P\int_c \frac{2iz\,\frac{dz}{iz}}{26zi+5z^2-5} = \int_c \frac{2}{5z^2+26zi-5}\,dz$$

$$= R.P\int \frac{2\,dz}{(z+\frac{i}{5})(z+5i)}$$

$5z^2+26zi-5$

$a=5$ $b=26i$ $c=-5$ ~~-25~~ ~~$-26i$~~

$\frac{-26i \pm \sqrt{(26i)^2-4(5)(-5)}}{10}$

$\frac{-26i \pm \sqrt{-676+100}}{10}$

$\frac{-26i \pm \sqrt{-576}}{10}$

$\frac{-26i \pm 24i}{10}$

$-\frac{i}{5}, -5i$

let $f(z) = \frac{1}{(z+\frac{i}{5})(z+5i)}$

$z = -\frac{i}{5}$ is a simple pole

$z=-5i$ is a simple pole

$z = -\frac{i}{5}$ lies inside c

$z = -5i$ lies outside c

$$\operatorname{Res}\left[f(z), -\tfrac{i}{5}\right] = \lim_{z\to -\frac{i}{5}} \cancel{(z+\tfrac{i}{5})}\frac{1}{\cancel{(z+\frac{i}{5})}(z+5i)}$$

$$= \lim_{z\to -\frac{i}{5}} \frac{1}{z+5i} = \frac{1}{-\frac{i}{5}+5i} = \frac{5}{24i}$$

By cauchy Residue theorem

$$\int_c f(z)dz = 2\pi i \left[\frac{5}{24i}\right]$$
$$= \frac{5\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = 2\left[\frac{5\pi}{12}\right]$$
$$= \frac{5\pi}{6}$$

③ Evaluate using contour integration $\int_0^{2\pi} \frac{d\theta}{5-\sin\theta} = \pi/6$

④ Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$

let $z = e^{i\theta}$ $\qquad \cos\theta = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$dz = ie^{i\theta}d\theta$ $\qquad \sin\theta = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

$dz = iz\,d\theta$

$d\theta = \frac{dz}{iz}$

$z^2 = (e^{i\theta})^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

R.P of $e^{i2\theta} = \cos 2\theta$

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5-4\cos\theta} = R.P\int_c \frac{z^2 \frac{dz}{iz}}{5-4\left[\frac{1}{2}\left[\frac{z^2+1}{z}\right]\right]}$$

$$= R.P\int_c \frac{z^2 \frac{dz}{iz} \times z}{5z-2z^2-2}$$

$$= R.P \frac{-1}{i}\int_c \frac{z^2\,dz}{2z^2-5z+2}$$

let $f(z) = \frac{z^2}{2z^2-5z+2} = \frac{z^2}{(z-2)(2z-1)}$

$z=2$ is a simple pole lies outside C

$z=1/2$ is a simple pole lies inside C

$$Res\left[f(z), \tfrac{1}{2}\right] = \lim_{z\to 1/2} \left[z-\tfrac{1}{2}\right] \frac{z^2}{[z-2][2z-1]}$$

$$= \frac{1}{2}\lim_{z\to 1/2} (2z-1) \frac{z^2}{(z-2)(2z-1)} = \frac{1}{2}\lim_{z\to 1/2} \frac{1/4}{(z-2)} = \frac{1}{2}\,\frac{1/4}{1/2-2}$$

$$= \left(\tfrac{1}{2}\right)\frac{1/4}{\left(-\frac{3}{2}\right)}$$

$$= \left(\tfrac{1}{4}\right)\tfrac{1}{2}\left[\tfrac{-2}{3}\right] = \tfrac{-1}{12}$$

$Res[f(z), \frac{1}{2}] = \frac{-1}{12}$

Hence Cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i\,[\text{Sum of the residues}]$$

$$= 2\pi i \left[\frac{-1}{12}\right] = \frac{-\pi i}{6}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\,d\theta = R\cdot P\, -\frac{1}{\cancel{i}}\left[\frac{-\pi \cancel{i}}{6}\right]$$

$$= R\cdot P\ \pi/6 = \pi/6$$

⑤ Evaluate the integral $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$

let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$ $\qquad \cos\theta = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$d\theta = \frac{dz}{iz}$ $\qquad \sin\theta = \frac{1}{2}\left[\frac{z^2-1}{z}\right]$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= R\cdot P\ \frac{1-z^2}{2}$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{5+4\cos\theta} = \int_c \frac{\frac{1-z^2}{2}\,\frac{dz}{iz}}{5+\cancel{4}^{2}\left[\frac{1}{\cancel{2}}\left[\frac{z^2+1}{z}\right]\right]}$$

$$= R\cdot P\,\frac{1}{2}\int_c \frac{\frac{1-z^2}{i\cancel{z}} \times \cancel{z}\,dz}{5z+2z^2+2}$$

$$= R\cdot P\,\frac{1}{2i}\int_c \frac{1-z^2\,dz}{2z^2+5z+2}$$

$$\text{Let } f(z) = \frac{1-z^2}{2z^2+5z+2} = \frac{1-z^2}{(z+2)(2z+1)}$$

$z=-2$ is a simple pole which lies outside C

$z = -\frac{1}{2}$ is a simple pole which lies inside C

$$Res\left[f(z)\, -\tfrac{1}{2}\right] = \lim_{z\to -\frac{1}{2}} \left[z+\tfrac{1}{2}\right] f(z)$$

$$= \lim_{z \to -1/2} \frac{\cancel{2z+1}}{2} \frac{1-z^2}{(z+2)(\cancel{2z+1})}$$

$$= \frac{1}{2} \lim_{z \to -1/2} \frac{1-z^2}{z+2}$$

$$= \frac{1}{2} \frac{1-\frac{1}{4}}{-\frac{1}{2}+2} = \frac{1}{2} \frac{(3/4)}{(3/2)} = \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{4}$$

Hence by cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i \left[\frac{1}{4}\right]$$

$$= \frac{\pi i}{2}$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta = R.P \frac{1}{2i}\left[\frac{\pi i}{2}\right]$$

$$= R.P \ \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

⑥ Evaluate $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$ $a>|b|$

Soln:

if $f(2a-x) = f(x)$ then $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

Here $\cos(2\pi-\theta) = \cos\theta$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$$

(ie) $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ —— ①

Let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$ $\qquad \cos\theta = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$d\theta = \frac{dz}{iz}$ $\qquad \sin\theta = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_c \frac{\frac{dz}{iz}}{a+b\left(\frac{1}{2}\left(\frac{z^2+1}{z}\right)\right)} = \frac{1}{2}\int_c \frac{\frac{dz}{i\cancel{z}} \times 2\cancel{z}}{2az+bz^2+b}$$

$$= \frac{1}{\cancel{2}i} \times \cancel{2} \int_c \frac{dz}{bz^2+2az+b}$$

$$= \frac{1}{bi} \int_c \frac{dz}{z^2+\frac{2a}{b}z+1} \quad \text{—②}$$

$$z^2+\frac{2a}{b}z+1$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$z = \frac{-\frac{2a}{b} \pm 2\frac{\sqrt{a^2-b^2}}{b}}{2}$$

$$z = \frac{-\frac{\cancel{2}a}{b} \pm \cancel{2}\frac{\sqrt{a^2-b^2}}{b}}{\cancel{2}}$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

$$\alpha-\beta = \frac{-a+\sqrt{a^2-b^2}-(-a-\sqrt{a^2-b^2})}{b}$$

$$= \frac{-\cancel{a}+\sqrt{a^2-b^2}+\cancel{a}+\sqrt{a^2-b^2}}{b}$$

$$= \frac{2\sqrt{a^2-b^2}}{b}$$

Given $a > |b|$

Let $a=2 \quad b=1$

$$z = \frac{-2+\sqrt{4-1}}{1} = -2+\sqrt{3} = -2+1.732 = -0.268$$

lies inside c

$$z = \frac{-2-\sqrt{4-1}}{1} = -2-\sqrt{3} = -2-1.732 = -3.732$$

lies outside c

$\therefore z = \alpha$ lies inside C

$$\text{Res}[f(z), \alpha] = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)\,dz = \cancel{2}\pi i\left[\frac{b}{\cancel{2}\sqrt{a^2-b^2}}\right] = \frac{b\pi i}{\sqrt{a^2-b^2}}$$

$\therefore 2 \Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{bi}\left[\frac{2\pi i}{\sqrt{a^2-b^2}}\right]$

$= \frac{\pi}{\sqrt{a^2-b^2}}$ //

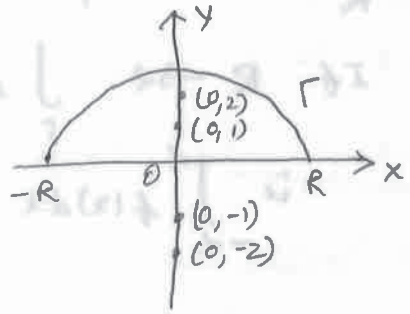
Type:-2

Problems based on contour integration $\int_{-\alpha}^{\alpha} \frac{P(x)}{Q(x)}\,dx$

① Evaluate $\int_{-\alpha}^{\alpha} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$ using contour integration.

Soln: Let us consider $\int_C f(z)\,dz = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz$

where C consists of the semi-circle $\Gamma : |z| = R$ and the bounding diameter $[-R, R]$



Now

$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$

The poles of $f(z)$ are the solutions of

$(z^2+1)(z^2+4) = 0$

$z = i, -i, 2i, -2i$

$z = i$ is a simple pole lies inside Γ

$z = 2i$ is a simple pole lies inside Γ

$z = -i$ is a simple pole lies outside Γ

$z = -2i$ is a simple pole lies outside Γ

$R_1[f(z), i] = \lim_{z\to i} (z-i)\,f(z)$

$= \lim_{z\to i} \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z^2+4)}$

$= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)}$

$= \frac{i^2}{(i+i)(i^2+4)} = \frac{-1}{2i(-1+4)} = \frac{-1}{6i}$

$$R_2\left[f(z), 2i\right] = \lim_{z\to 2i} (z-2i)\, f(z)$$

$$= \lim_{z\to 2i} \cancel{(z-2i)} \frac{z^2}{(z^2+1)(z+2i)\cancel{(z-2i)}}$$

$$= \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{(2i)^2}{((2i)^2+1)(2i+2i)} = \frac{-4}{(-4+1)(4i)} = \frac{-1}{-3i} = \frac{1}{3i}$$

Here by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\left[R_1 + R_2\right]$$

$$= 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = 2\pi i\left[\frac{-1+2}{6i}\right] = \frac{\pi}{3}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = \frac{\pi}{3}$$

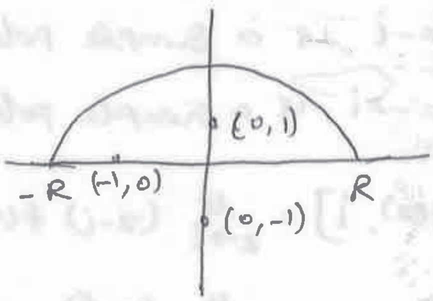
If $R \to \infty$ $\int_\Gamma f(z)\,dz \to 0$

i.e $\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3}$ Hence $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}$

② Using contour integration evaluate $\int_{-\infty}^{\infty} \frac{x\,dx}{(x+1)(x^2+1)}$

Soln:
Let us consider $\int_C f(z)\,dz = \int_C \frac{z\,dz}{(z+1)(z^2+1)}$

where C consists of the semi-circle $\Gamma: |z| = R$ and the bounding diameter $[-R, R]$



Now $\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$

$(z+1)(z^2+1) = 0$

$z = -1$ is a simple pole lies ~~outside~~ inside Γ

$z = i$ is a simple pole lies inside Γ

$z = -i$ is a simple pole lies ~~inside~~ outside Γ

$$R_1\left[f(z), i\right] = \lim_{z\to i} (z-i)\, f(z) = \lim_{z\to i} \cancel{(z-i)} \frac{z}{(z+1)(z+i)\cancel{(z-i)}}$$

$$= \lim_{z\to i} (z-i) \frac{z}{(z+1)(z^2+1)}$$

$$\underset{z\to i}{lt} \frac{z}{(z+1)(z+i)} = \frac{\cancel{i}}{(i+1)(2\cancel{i})} = \frac{1}{2(1+i)}$$

$$R_2[f(z),-1] = \underset{z\to -1}{lt} \cancel{(z+1)} \frac{z}{\cancel{(z+1)}(z^2+1)} = \frac{-1}{1+1} = -\frac{1}{2}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[R_1] + \pi i[R_2] \rightarrow [\text{sum of the residues of the real axis}]$$

$$= \cancel{2}\pi i\left[\frac{1}{\cancel{2}(1+i)}\right] + \pi i\left[-\frac{1}{2}\right]$$

$$= \pi i\left[\frac{1}{1+i}\right] + \pi i\left[-\frac{1}{2}\right]$$

$$= \pi i\left[\frac{1}{1+i} - \frac{1}{2}\right]$$

$$= \pi i\left[\frac{2-1-i}{2(1+i)}\right] = \frac{\pi i}{2}\left[\frac{1-i}{1+i}\right] = \frac{\pi i}{2}(-i) = \frac{\pi}{2}$$

If $R\to \alpha$ $\int_\Gamma f(z)dz \to 0$

$$\therefore \int_{-\alpha}^{\alpha} f(x)\,dx = \frac{\pi}{2}$$

Hence $\displaystyle\int_{-\alpha}^{\alpha} \frac{x\,dx}{(x+1)(x^2+1)} = \frac{\pi}{2}$ //

(3) Show that $\displaystyle\int_{-\alpha}^{\alpha} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Soln

Let us consider $\displaystyle\int_c f(z)dz = \int_c \frac{dz}{(z^2+1)^3}$

where c is the semi circle Γ with the bounding diameter $[-R, R]$

$$\int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$(z^2+1)^3 = 0$$

$z=i$ is a pole of order 3 lies inside Γ

$z=-i$ is a pole of order 3 lies outside Γ

$$\text{Res}[f(z), i] = \underset{z\to i}{lt} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-i)^3 f(z)\right]$$

$$= \frac{1}{2} \underset{z\to i}{lt} \frac{d^2}{dz^2}\left[\cancel{(z-i)^3} \frac{1}{(z+i)^3 \cancel{(z-i)^3}}\right]$$

$$= \frac{1}{2} \underset{z\to i}{lt} \frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]$$

$= \frac{1}{2} \lim_{z \to i} \frac{d}{dz} \left[\frac{-3}{(z+i)^4} \right]$

$= \frac{1}{2} \lim_{z \to i} \left[\frac{12}{(z+i)^5} \right]$

$= \frac{1}{2} \left[\frac{12}{(2i)^5} \right] = \frac{6}{2^5 i^5} = \frac{3}{16i}$

$\therefore$ By Cauchy's residue theorem

$\int_c f(z)\,dz = \cancel{2}\pi \cancel{i} \left[\frac{3}{\cancel{16}_8 \cancel{i}} \right] = \frac{3\pi}{8}$

If $R \to \alpha$, $|z| \to \alpha$, $\int_\Gamma f(z)\,dz \to 0$

$\therefore \int_{-\alpha}^{\alpha} f(x)\,dx = \frac{3\pi}{8}$

$\int_{-\alpha}^{\alpha} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

(4) Evaluate $\int_{-\alpha}^{\alpha} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx$

Let $\int_c f(z)\,dz = \int_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz$

Where C is the upper half of the Semi-circle Γ with the bounding diameter $[-R, R]$

$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$

$z^4 + 10z^2 + 9 = 0$

put $t = z^2$ then $t^2 + 10t + 9 = 0$

$(t+9)(t+1) = 0$

$t = -9 \quad t = -1$

$z^2 = -9 \quad z^2 = -1$

$z = \pm 3i \quad z = \pm i$

out of these poles $z = i, 3i$ lies inside Γ

$\therefore$ Res $[f(z), i] = \lim_{z \to i} (z-i) f(z)$

$$= \lim_{z \to i} (z-i) \frac{z^2+2-z}{z^4+10z^2+9} = 0/0$$

$$= \lim_{z \to i} \frac{(z-i)(2z-1) + (z^2-z+2)(1)}{4z^3+20z}$$

$$= \lim_{z \to i} \frac{(z-i)(2z-1) + (z^2-z+2)}{4z[z^2+5]}$$

$$= \frac{(i-i)(2i-1) + (i^2-i+2)}{4i[i^2+5]}$$

$$= \frac{0 + (-1+2-i)}{4i[-1+5]} = \frac{1-i}{4i[4]} = \frac{1-i}{16i}$$

Res $[f(z), 3i] = \lim_{z \to 3i} (z-3i) f(z)$

$$= \lim_{z \to 3i} (z-3i) \frac{z^2-z+2}{z^4+10z^2+9} = 0/0 \text{ form}$$

$$= \lim_{z \to 3i} \frac{(z-3i)(2z-1) + (z^2-z+2)(1)}{4z^3+20z}$$

$$= \lim_{z \to 3i} \frac{(z-3i)(2z-1) + (z^2-z+2)}{4z[z^2+5]}$$

$$= \frac{0 + ((3i)^2-3i+2)}{4(3i)[(3i)^2+5]} = \frac{(-9+2-3i)}{12i[-9+5]} = \frac{-7-3i}{-48i} = \frac{\cancel{-}[7+3i]}{\cancel{-}48i}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i \left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= \frac{2\pi \cancel{i}}{16 \cancel{i}} \left[1-i + \frac{7+3i}{3}\right]$$

$$= \pi/8 \left[\frac{3-\cancel{3i}+7+\cancel{3i}}{3}\right]$$

$$= \frac{\pi}{\cancel{8}_4} \left[\frac{\cancel{10}^5}{3}\right]$$

$$= \frac{5\pi}{12}$$

If $R \to \infty$ $\int_\Gamma f(z)dz \to 0$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{5\pi}{12}$$

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}$$

⑤ prove that $\int_{-\alpha}^{\alpha} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}, \quad a>b>0$

Soln:

Consider $\int_c f(z)dz = \int_c \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$ where c is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$

$$\int_c f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)dz$$

$z^2+a^2=0 \qquad z^2+b^2=0$

$z=\pm ai \qquad z=\pm bi$

$z=ai, bi$ lies inside Γ and also a simple pole

$z=-ai, -bi$ lies outside Γ and also a simple pole

$$\text{Res}[f(z), ai] = \lim_{z\to ai}(z-ai)f(z)$$

$$= \lim_{z\to ai}\cancel{(z-ai)}\frac{z^2}{\cancel{(z-ai)}(z+ai)(z^2+b^2)}$$

$$= \frac{(ai)^2}{(2ai)((ai)^2+a^2)} = \frac{-a^2}{(2ai)(-a^2+b^2)} = \frac{-a}{(2i)(-a^2+b^2)}$$

$$= \frac{-a}{2i(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}$$

$$\text{Res}[f(z), bi] = \lim_{z\to bi}\cancel{(z-bi)}\frac{z^2}{(z^2+a^2)(z+bi)\cancel{(z-bi)}}$$

$$= \frac{(bi)^2}{(2bi)((bi)^2+a^2)} = \frac{-b^2}{2bi(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$$

By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[R_1+R_2]$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}[a-b] = \frac{\pi}{(a+b)(a-b)}(a-b)$$

$$= \frac{\pi}{a+b}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi}{a+b}$$

If $R \to \alpha$ then $\int_{\Gamma} f(z)\,dz = 0$

$$\int_{-\alpha}^{\alpha} f(x)\,dx = \frac{\pi}{a+b}$$

$$\int_{-\alpha}^{\alpha} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} \text{ //}$$

6. Evaluate $\int_0^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a>0, b>0$

$$\int_{-\alpha}^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\int_0^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$\int_0^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$\int_c f(z)\,dz = \int_c \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

Where c is the semi-circle Γ with the bounding diameter $[-R, R]$

By cauchy's residue theorem we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

$z^2+a^2=0 \qquad z^2+b^2=0$

$z=\pm ai \qquad z=\pm bi$

$z = ai, bi$ is a simple pole lies inside Γ

$z=-ai, -bi$ is a simple pole lies outside Γ

$$\text{Res}[f(z), ai] = \lim_{z\to ai} \cancel{(z-ai)} \frac{1}{\cancel{(z-ai)}(z+ai)(z^2+b^2)}$$

$$= \frac{1}{2ai(-a^2+b^2)} = \frac{1}{2ai(b^2-a^2)}$$

$$\operatorname{Res}[f(z), bi] = \lim_{z \to bi} (z-bi) \frac{1}{(z^2+a^2)(z-bi)(z+bi)}$$

$$= \frac{1}{-b^2+a^2(2bi)} = \frac{1}{(2bi)(a^2-b^2)}$$

Hence by cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i \left[\frac{1}{(2ai)(b^2-a^2)} + \frac{1}{(2bi)(a^2-b^2)}\right]$$

$$= 2\pi i \left[\frac{-1}{(2ai)(a^2-b^2)} + \frac{1}{(2bi)(a^2-b^2)}\right]$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2i}(a^2-b^2)}\left[\frac{-1}{a} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{-b+a}{ab}\right]$$

$$= \frac{\pi}{(a+b)\cancel{(a-b)}}\left[\frac{\cancel{a-b}}{ab}\right]$$

$$= \frac{\pi}{ab(a+b)}$$

(iii) $$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi}{ab(a+b)}$$

If $R \to \alpha$ then $\int_{\Gamma} f(z)\,dz \to 0$

$$\int_{-\alpha}^{\alpha} f(x)\,dx = \frac{\pi}{ab(a+b)}$$

$$\int_0^{\alpha} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\left[\frac{\pi}{ab(a+b)}\right] = \frac{\pi}{2ab(a+b)} \text{ //}$$

(7) Evaluate $\int_0^{\alpha} \frac{dx}{x^4+10x^2+9}$.

$$\int_{-\alpha}^{\alpha} \frac{dx}{x^4+10x^2+9} = 2\int_0^{\alpha} \frac{dx}{x^4+10x^2+9}$$

$$\int_0^{\alpha} \frac{dx}{x^4+10x^2+9} = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{dx}{x^4+10x^2+9}$$

$$\int_C f(z)dz = \int_C \frac{dz}{z^4+10z^2+9}$$

(b) $$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$z^4+10z^2+9=0$$

$t=z^2$ $\quad$ $t=-9$ $\quad$ $t=-1$

$t^2+10t+9=0$ $\quad$ $z^2=-9$ $\quad$ $z^2=-1$

$(t+9)(t+1)=0$ $\quad$ $z=\pm 3i$ $\quad$ $z=\pm i$

$z=i, 3i$ is a simple pole lies inside Γ

$z=-i, -3i$ is a simple pole lies outside Γ

$$\text{Res}[f(z), i] = \lim_{z\to i} \cancel{(z-i)} \frac{1}{(z^2+9)(z+i)\cancel{(z-i)}}$$

$$= \frac{1}{(i^2+9)(2i)} = \frac{1}{(-1+9)2i} = \frac{1}{16i}$$

$$\text{Res}[f(z), 3i] = \lim_{z\to 3i} \cancel{(z-3i)} \frac{1}{(z+3i)\cancel{(z-3i)}(z^2+1)} = \frac{1}{6i\,((3i)^2+1)}$$

$$= \frac{1}{(6i)(9i^2+1)}$$

$$= \frac{1}{(6i)(-9+1)} = \frac{1}{(6i)(8)} = \frac{-1}{48i}$$

By Cauchy residue theorem

$$\int_C f(z)dz = 2\pi i\,[R_1+R_2]$$

$$= 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right]$$

$$= \frac{2\pi i}{16i}\left[1 - \frac{1}{3}\right]$$

$$= \frac{\pi}{8}\left[\frac{2}{3}\right] = \frac{\pi}{12}$$

$$\therefore \int_0^\infty \frac{dx}{x^4+10x^2+9} = \frac{1}{2}\left[\frac{\pi}{12}\right] = \frac{\pi}{24}$$

(ii) $$\int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz = \frac{\pi}{12}$$

If $R\to\infty$ then $\int_\Gamma f(z)dz \to 0$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{12}$$

Type III

Problems based on contour integration

$$\int_{-\alpha}^{\alpha} f(x)\cos nx\,dx \quad (\text{or}) \quad \int_{-\alpha}^{\alpha} f(x)\sin nx\,dx$$

① Evaluate $\int_0^{\alpha} \frac{\cos ax}{x^2+1}\,dx,\ a>0$

Soln :-

$$2\int_0^{\alpha} \frac{\cos ax}{x^2+1}\,dx = \int_{-\alpha}^{\alpha} \frac{\cos ax}{x^2+1}\,dx$$

$$\int_0^{\alpha} \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{\cos ax}{x^2+1}\,dx$$

Let $\int_C f(z)\,dz = R.P \int \frac{e^{iaz}}{z^2+1}$

where C is the upper half of the semi-circle Γ with the bounding diameter $[-R,R]$

By Cauchy's residue theorem we have

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

The poles of $f(z)$ are the solutions of $z^2+1=0$

$$z = \pm i$$

$z=i$ is a simple pole lies inside Γ

$z=-i$ is a simple pole lies out side Γ

$$\therefore \text{Res}[f(z), i] = \lim_{z\to i} (z-i)\,f(z)$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{e^{iaz}}{(z+i)\cancel{(z-i)}}$$

$$= \lim_{z\to i} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Hence $\int_C f(z)\,dz = R.P\ 2\pi i\,[\text{sum of the Residues}]$

$$= R.P\ \cancel{2}\pi\cancel{i}\left[\frac{e^{-a}}{\cancel{2i}}\right]$$

$$= R.P\ \pi e^{-a} = \pi e^{-a}$$

i.e.) $\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \pi e^{-a}$

If $R \to \infty$ then $\int_{\Gamma} f(z)\,dz \to 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$$

$$\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \tfrac{1}{2}\left[\pi e^{-a}\right] = \tfrac{\pi}{2} e^{-a}$$

(2) Show that $\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi/2$

$$2\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

$$= \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \quad \text{—— (1)}$$

To find $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

Consider $\int_{c} f(z)\,dz = IP \int_{c} \frac{e^{iz}}{z}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$

$$\int_{c} f(z)\,dz = IP \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

$z = 0$ is a simple pole lies on the real axis inside Γ

$$\therefore \operatorname{Res}[f(z), 0] = \lim_{z\to 0} (z-0) f(z)$$

$$= \lim_{z\to 0} z\left[\frac{e^{iz}}{z}\right] = \lim_{z\to 0} e^{iz} = e^{0} = 1$$

Hence by Cauchy's residue theorem

$$\int_{c} f(z)\,dz = 2\pi i(0) + \pi i(1)$$

$$= IP\,\pi i$$

$$IP \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = IP\,\pi i$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi i}{i} = \int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi/2$$

LAPLACE TRANSFORM

Definition:

The Laplace transform of a continuous function $f(t)$ is defined by $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt,\ t>0$.

Important results:

1. $L(1) = 1/s \quad s>0$

$L(1) = 1/s$ $\quad L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$

2. $L(t^n) = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2 \ldots$

$L(t) = 1/s^2$ $\quad (n-1)!$

3. $L(t^n) = \frac{\Gamma n+1}{s^{n+1}}$ where n is not an integer & $\Gamma n = \cancel{n\Gamma n-1}$

4. $L(e^{at}) = \frac{1}{s-a}$

5. $L(e^{-at}) = \frac{1}{s+a}$

6. $L(\sin at) = \frac{a}{s^2+a^2}$

7. $L(\cos at) = \frac{s}{s^2+a^2}$

8. $L(\sinh at) = \frac{a}{s^2-a^2}$

9. $L(\cosh at) = \frac{s}{s^2-a^2}$

10. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

11. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

12. $L(c\,f(t)) = c\,L\,f(t)$

13. $\sinh x = \frac{e^x - e^{-x}}{2}$

14. $\cosh x = \frac{e^x + e^{-x}}{2}$

$L($

Problems

① Find $L(t^3)$

Soln: WKT $L(t^n) = \frac{n!}{s^{n+1}}$

$$L(t^3) = \frac{3!}{s^{3+1}}$$

$$= \frac{3!}{s^4} = \frac{6}{s^4}$$

② Find $L(\sqrt{t})$

Soln:

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

WKT $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$$L(t^{1/2}) = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} = \frac{1}{2}\,\frac{\sqrt{\pi}}{s^{3/2}}$$

③ Find $L(e^{3t})$

Soln:

WKT $L(e^{at}) = \dfrac{1}{s-a}$

$$L(e^{3t}) = \frac{1}{s-3}$$

④ Find $L(e^{-4t})$

Soln:

WKT $L(e^{-at}) = \dfrac{1}{s+a}$

$$L(e^{-4t}) = \frac{1}{s+4}$$

⑤ Find $L(\sin 4t)$

Soln:

WKT $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$L(\sin 4t) = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$$

⑥ Find $L(\cos 7t)$

Soln:

WKT $L(\cos at) = \dfrac{s}{s^2+a^2}$

$$L(\cos 7t) = \frac{s}{s^2+7^2} = \frac{s}{s^2+49}$$

⑦ Find $L(\sinh 3t)$

Soln:

WKT $L(\sinh at) = \dfrac{a}{s^2-a^2}$

$$L(\sinh 3t) = \frac{3}{s^2-3^2} = \frac{3}{s^2-9}$$

⑧ Find $L(\cosh 2t)$

Soln:

WKT $L(\cosh at) = \dfrac{s}{s^2-a^2}$

$$L(\cosh 2t) = \frac{s}{s^2-2^2} = \frac{s}{s^2-4}$$

⑨ Find $L(\sin^2 2t)$

$$\boxed{\sin^2 A = \frac{1-\cos 2A}{2}}$$

$$L(\sin^2 2t) = L\left[\frac{1-\cos 2(2t)}{2}\right]$$

$$= L\left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

⑩ Find $L(\sin 5t \cos 2t)$

Soln:

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$$L(\sin 5t \cos 2t) = L\left[\frac{\sin 7t + \sin(3t)}{2}\right]$$

$$= L\left[\frac{\sin 7t + \sin 3t}{2}\right]$$

$$= \frac{1}{2}\left[L(\sin 7t) + L(\sin 3t)\right]$$

$$= \frac{1}{2}\left[\frac{7}{s^2+7^2} + \frac{3}{s^2+3^2}\right]$$

$$= \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

(11) Find $L[(\sin t - \cos t)^2]$

Soln:

$= L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$

$= L[1 - 2\sin t \cos t]$

$= L[1 - \sin 2t]$

$= L[1] - L[\sin 2t]$

$= \frac{1}{s} - \frac{2}{s^2+2^2}$

$= \frac{1}{s} - \frac{2}{s^2+4}$

(12) Find $L[8e^{8t} + \cosh 3t + \sin 5t]$

Soln:

$L[8e^{8t} + \cosh 3t + \sin 5t] = 8L[e^{8t}] + L[\cosh 3t] + L[\sin 5t]$

$= 8\frac{1}{s-8} + \frac{s}{s^2-3^2} + \frac{5}{s^2+5^2}$

$= \frac{8}{s-8} + \frac{s}{s^2-9} + \frac{5}{s^2+25}$

(13) Find $L[\cos^4 t]$

Soln:

$L[\cos^2 t]^2 \Rightarrow L\left[\frac{1+\cos 2t}{2}\right]^2$

$\Rightarrow L\left[\frac{(1+\cos 2t)^2}{4}\right]$

$= \frac{1}{4} L[1 + \cos^2 2t + 2\cos 2t]$

$= \frac{1}{4} L\left[1 + \frac{1+\cos 4t}{2} + 2\cos 2t\right]$

$= \frac{1}{4} L\left[\frac{2 + 1 + \cos 4t + 2\cos 2t}{2}\right]$

$= \frac{1}{4} L\left[\frac{3 + \cos 4t + 2\cos 2t}{2}\right]$

$= \frac{1}{8} L[3 + \cos 4t + 2\cos 2t]$

$= \frac{1}{8}\left[3L[1] + L[\cos 4t] + 2L[\cos 2t]\right]$

$= \frac{1}{8}\left[3\left(\frac{1}{s}\right) + \frac{s}{s^2+4^2} + 2\left(\frac{s}{s^2+2^2}\right)\right]$

$= \frac{1}{8}\left[\frac{3}{s} + \frac{s}{s^2+16} + \frac{2s}{s^2+4}\right]$

(14) Find $L[\sin^2 t \cos^3 t]$

$\cos 3A = 4\cos^3 A - 3\cos A$

$4\cos^3 A = 3\cos A + \cos 3A$

$\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

soln:

$$L[\sin^2 t \cos^3 t] = L\left[\left(\frac{1-\cos 2t}{2}\right)\left(\frac{3\cos t + \cos 3t}{4}\right)\right]$$

$$= \frac{1}{8} L\left[(1-\cos 2t)(3\cos t + \cos 3t)\right]$$

$$= \frac{1}{8} L\left[3\cos t + \cos 3t - 3\cos t\cos 2t - \cos 2t\cos 3t\right]$$

$$= \frac{1}{8} L\left[3\cos t + \cos 3t - \frac{3}{2}[\cos 3t + \cos t] - \frac{1}{2}[\cos 5t + \cos t]\right]$$

$$= \frac{1}{8} L\left[3\cos t + \cos 3t - \frac{3}{2}\cos 3t - \frac{3}{2}\cos t - \frac{1}{2}\cos 5t - \frac{1}{2}\cos t\right]$$

$$= \frac{1}{8} L\left[-\frac{1}{2}\cos 3t - \frac{1}{2}\cos 5t + \cos t\right]$$

$$= \frac{1}{16} L\left[2\cos t - \cos 3t - \cos 5t\right]$$

$$= \frac{1}{16}\left[2\left(\frac{s}{s^2+1^2}\right) - \frac{s}{s^2+3^2} - \frac{s}{s^2+5^2}\right]$$

$$= \frac{1}{16}\left[\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right]$$

(15) Find the Laplace transform of $L[\sin^3 2t + \cosh^2 3t]$

soln:

$$L[\sin^3 2t + \cosh^2 3t] = L\left[\frac{3\sin 2t - \sin 6t}{4} + \left(\frac{e^{3t} + e^{-3t}}{2}\right)^2\right]$$

$$= L\left[\frac{3\sin 2t - \sin 6t}{4} + \frac{1}{4}\left(e^{6t} + e^{-6t} + 2\right)\right]$$

$$= \frac{1}{4} L\left[3\sin 2t - \sin 6t + e^{6t} + e^{-6t} + 2\right]$$

$$= \frac{1}{4}\left[3L(\sin 2t) - L(\sin 6t) + L(e^{6t}) + L(e^{-6t}) + 2L(1)\right]$$

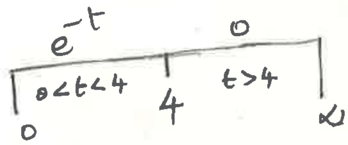
$$= \frac{1}{4}\left[3\left(\frac{2}{s^2+2^2}\right) - \frac{6}{s^2+6^2} + \frac{1}{s-6} + \frac{1}{s+6} + 2\left(\frac{1}{s}\right)\right]$$

$$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36} + \frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$$

Linearity property:

$$L\left(f(t)+g(t)\right) = L\left(f(t)\right) + L\left(g(t)\right)$$

① Find $L(f(t))$ if $f(t) = \begin{cases} e^{-t}, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$



Soln:

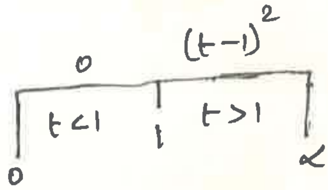
$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{4} e^{-st} e^{-t}\,dt + 0$$

$$= \int_0^{4} e^{-(s+1)t}\,dt$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4 = \left[\frac{e^{-(s+1)4}}{-(s+1)} - \frac{1}{-(s+1)}\right] = \frac{1-e^{-(s+1)4}}{(s+1)}$$

② Find $L\,f(t)$ if $f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & t < 1 \end{cases}$



Soln:

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= 0 + \int_1^{\infty} e^{-st}(t-1)^2\,dt$$

$\int u\,dv = uv - u'v_1 + u''v_2 + \dots$

$u = (t-1)^2$, $dv = e^{-st}dt$

$u' = 2(t-1)$, $v = \frac{e^{-st}}{-s}$

$u'' = 2$, $v_1 = \frac{e^{-st}}{s^2}$

$v_2 = \frac{e^{-st}}{-s^3}$

By using Bernoulli's theorem

$$= \int_1^{\infty} e^{-st}(t-1)^2\,dt$$

$$= \left[(t-1)^2\left(\frac{e^{-st}}{-s}\right) - 2(t-1)\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_1^{\infty}$$

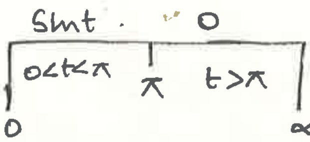
$$= \left[-(t-1)^2\frac{e^{-st}}{s} - 2(t-1)\frac{e^{-st}}{s^2} - 2\left(\frac{e^{-st}}{s^3}\right)\right]_1^{\infty}$$

$e^{-\infty} = 0$

$$= \left[(0-0-0) - \left(0-0-2\left(\frac{e^{-s}}{s^3}\right)\right)\right]$$

$$= \frac{2}{s^3}e^{-s}$$

③ Find $L[f(t)]$ if $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$



Soln:

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{\pi} e^{-st} \sin t \, dt + 0 \qquad \left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) \right]$$

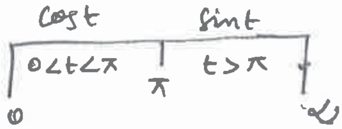
Put $a = -s \quad b = 1 \quad x = t$

$$\therefore L\{f(t)\} = \left[\frac{e^{-st}}{s^2+1}(-s \sin \cancel{t} - \cos t) \right]_0^{\pi}$$

$$= \left[\frac{e^{-s\pi}}{s^2+1}(-s \sin \pi - \cos \pi) + \frac{1}{s^2+1} \right]$$

$$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1 + e^{-\pi s}}{s^2+1} //$$

④ Find $L[f(t)]$ ~~if~~ $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$



Soln:- $L\, f(t) = \int_0^{\infty} e^{-st} f(t) \, dt$

$$\left[\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) \right]$$

$$= \int_0^{\pi} e^{-st} \cos t \, dt + \int_{\pi}^{\infty} e^{-st} \sin t \, dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s \cos t + \sin t) \right]_0^{\pi} + \left[\frac{e^{-st}}{s^2+1}(-s \sin t - \cos t) \right]_{\pi}^{\infty}$$

$$= \left[\frac{e^{-s\pi}}{s^2+1}(-s \cos \pi + \sin \pi) + \frac{s}{s^2+1} \right] + \left[0 - \frac{e^{-s\pi}}{s^2+1} \right]$$

$$= \frac{s e^{-s\pi}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-s\pi}}{s^2+1}$$

$$= \frac{1}{s^2+1} \left[s e^{-s\pi} + s - e^{-s\pi} \right]$$

$$= \frac{1}{s^2+1} \left[s + e^{-s\pi}(s-1) \right] //$$

First Shifting theorem.

Statement:

(i) If $L[f(t)] = \phi(s)$ then $L[e^{at} f(t)] = \phi(s-a)$

(ii) If $L[f(t)] = \phi(s)$ then $L[e^{-at} f(t)] = \phi(s+a)$

Proof:

(i) $L[f(t)] = \phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= \phi(s-a)$$

(ii) $L[f(t)] = \phi(s) = \int_0^\infty e^{-st} f(t) dt$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t) dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \phi(s+a)$$

problems:-

① Find $L[t^n e^{-at}]$

Soln:

$$L[t^n] = \frac{n!}{s^{n+1}}$$

$$\therefore L[t^n e^{-at}] = \left[\frac{n!}{s^{n+1}}\right]_{s \to s+a}$$

$$= \frac{n!}{(s+a)^{n+1}} \,//$$

② Find $L[e^{-at} \cos bt]$

$$L[\cos bt] = \frac{s}{s^2+b^2}$$

$$L[e^{-at} \cos bt] = \left[\frac{s}{s^2+b^2}\right]_{s \to s+a}$$

$$= \frac{s+a}{(s+a)^2+b^2} \,//$$

③ Find $L[e^{t/3} t^{-1/2}]$

$$L[t^{-1/2}] = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L[e^{t/3} t^{-1/2}] = \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s \to (s-\frac{1}{3})} = \frac{\sqrt{\pi}}{\sqrt{(s-\frac{1}{3})}} \text{ //}$$

④ Find $L[e^{-t} t^9]$

$$L[t^9] = \frac{9!}{s^{9+1}}$$

$$L[e^{-t} t^9] = \left[\frac{9!}{s^{10}}\right]_{s \to (s+1)}$$

$$= \frac{9!}{(s+1)^{10}}$$

⑤ Find $L[e^{at} \sinh bt]$

$$L[\sinh bt) = \frac{b}{s^2-b^2}$$

$$L[e^{at} \sinh bt] = \left[\frac{b}{s^2-b^2}\right]_{s \to (s-a)}$$

$$= \frac{b}{(s-a)^2-b^2}$$

⑦ Find $L[t^2 e^{3t} \sin ht]$

$$L[t^2 \sinh t] = L\left[t^2\left(\frac{e^t - e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2} L\left[t^2 (e^t - e^{-t})\right]$$

$$= \frac{1}{2} L[t^2 e^t - t^2 e^{-t}]$$

$$L[t^2] = \frac{2}{s^3} \qquad L[e^t t^2] = \left[\frac{2}{s^3}\right]_{s \to (s-1)}$$

$$= \frac{2}{(s-1)^3}$$

$$L[e^t t^2] = \frac{2}{(s-1)^3} \qquad L[e^{-t} t^2] = \frac{2}{(s+1)^3}$$

$$\therefore L[t^2 \sinh t] = \frac{1}{2}\left[\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\right]$$

$$\therefore L[e^{3t}t^2 \sinh t] = \frac{1}{2}\left[\frac{2}{(s-1)^3} - \frac{2}{(s+1)^3}\right]_{s \to (s-3)}$$

$$= \frac{1}{2}\left[\frac{2}{(s-4)^3} - \frac{2}{(s-2)^3}\right]$$

$$= \frac{1}{(s-4)^3} - \frac{1}{(s-2)^3} \,//$$

(6) Find $L[e^{2t}(\cos 4t + 3\sin 4t)]$

Soln:

$$L[\cos 4t + 3\sin 4t] = L(\cos 4t) + 3L(\sin 4t)$$

$$= \left[\frac{s}{s^2+4^2} + 3\left(\frac{4}{s^2+4^2}\right)\right]$$

$$= \left[\frac{s}{s^2+16} + \frac{12}{s^2+16}\right]$$

$$L[e^{2t}(\cos 4t + 3\sin 4t)] = \left[\frac{s}{s^2+16} + \frac{12}{s^2+16}\right]_{s \to s-2}$$

$$= \left[\frac{s-2}{(s-2)^2+16} + \frac{12}{(s-2)^2+16}\right]$$

$$= \frac{s-2+12}{(s-2)^2+16} = \frac{s+10}{(s-2)^2+16} \,//$$

TRANSFORMS OF DERIVATIVES

(1) If $L[f(t)] = \phi(s)$ then $L[tf(t)] = -\frac{d}{ds}\phi(s) = -\phi'(s)$

Proof:-

$$\phi(s) = L[f(t)]$$

$$\frac{d}{ds}\phi(s) = \frac{d}{ds} L[f(t)]$$

$$= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \quad \text{(By defn)}$$

$$= \int_0^\infty \left[\frac{\partial}{\partial s} e^{-st}\right] f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^{\infty} e^{-st}\, t\, f(t)\, dt$$

$$= -L[t\, f(t)]$$

$$= -\frac{d}{ds}\phi(s)$$

$$\therefore L[t\, f(t)] = -\frac{d}{ds}\phi(s) = -\phi'(s)$$

Corollary:-

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\phi(s)$$

① Find $L[t \sin 2t]$

$$L[\sin 2t] = \frac{2}{s^2+4}$$

$$L[t \sin 2t] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2} \text{ //.}$$

② Find $L[t^2 e^{3t} \sinh t]$

$$L[\sinh t] = \frac{1}{s^2-1}$$

$$L[e^{3t}\sinh t] = \left[\frac{1}{s^2-1}\right]_{s \to s-3}$$

$$= \frac{1}{(s-3)^2-1}$$

$$L[t^2 e^{3t} \sinh t] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{(s-3)^2-1}\right]$$

$$= \frac{d}{ds}\left[\frac{0 - 2(s-3)(1)}{((s-3)^2-1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-2(s-3)}{((s-3)^2-1)^2}\right]$$

$$= -\left[\frac{[(s-3)^2-1]^2\, 2 - 2(s-3)\, 2[(s-3)^2-1]\, 2(s-3)}{[(s-3)^2-1]^4}\right]$$

$$= -\left[\frac{2[(s-3)^2-1]^2 - 8(s-3)^2[(s-3)^2-1]}{[(s-3)^2-1]^4}\right]$$

$$= -\cancel{[(s-3)^2-1]}\left[\frac{2[(s-3)^2-1] - 8(s-3)^2}{[(s-3)^2-1]^4}\right]$$

$$= \frac{-2[(s-3)^2-1] + 8(s-3)^2}{[(s-3)^2-1]^3}$$

$$= \frac{-2(s-3)^2+2+8(s-3)^2}{[s^2+9-6s-1]^3} = \frac{-2(s-3)^2+2+8(s-3)^2}{[s^2-6s+8]^3}$$

③ Find $L(te^{-t}\sin t)$

Soln:

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{0-2s}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

$$L[te^{-t}\sin t] = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to(s+1)}$$

$$= \left[\frac{2(s+1)}{[(s+1)^2+1]^2}\right]$$

④ Find $L[t\sin 3t\cos 2t]$

Soln:

$$L[\sin 3t\cos 2t] = \frac{1}{2}L[\sin 5t+\sin t]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= -\frac{1}{2}\left[\frac{5(-2s)}{(s^2+25)^2} + \frac{(-2s)}{(s^2+1)^2}\right]$$

$$= \left[\frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}\right]$$

⑤ Show that $\int_0^\infty e^{-t}\, t \sin t \, dt = \frac{1}{2}$

Soln:

By defn $\int_0^\infty e^{-st} f(t)\, dt = L[f(t)]$

$$\therefore \int_0^\infty e^{-t}\, t \sin t\, dt = \left[L(t \sin t)\right]_{s=1}$$

$$= \left[-\frac{d}{ds} L(\sin t)\right]_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]_{s=1}$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]_{s=1}$$

$$= \left[\frac{2s}{(s^2+1)^2}\right]_{s=1}$$

$$\int_0^\infty e^{-t}\, t \sin t\, dt = \frac{2}{(2)^2} = \frac{1}{2}$$

⑥ Show that $\int_0^\infty e^{-t}\, t \cos t\, dt = 0$

Soln:

By defn $\int_0^\infty e^{-st} f(t)\, dt = L[f(t)]$

$$\therefore \int_0^\infty e^{-t}\, t \cos t\, dt = L\left[(t \cos t)\right]_{s=1} = -\frac{d}{ds} L(\cos t)_{s=1}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=1}$$

$$= -\left[\frac{(s^2+1) - s(2s)}{(s^2+1)^2}\right]_{s=1} = -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]_{s=1}$$

$$= -\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s=1}$$

$$= -\left[\frac{1-1}{(1+1)^2}\right] = 0 //$$

INTEGRALS OF TRANSFORMS

If $L[f(t)] = \phi(s)$, and $\frac{1}{t} f(t)$ has a limit as $t \to 0$ then

$$L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} \phi(s)\, ds$$

problems

① Find $L\left[\frac{1-e^t}{t}\right]$

$\frac{0}{0}$ $\frac{\infty}{\infty}$ 0^0 ∞^0 indet

soln:

$$f(t) = 1-e^t$$

$$\frac{f(t)}{t} = \frac{1-e^t}{t}$$

$$\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{1-e^t}{t} = \frac{1-1}{0} = \frac{0}{0}$$

Apply L' Hospitals Rule

$$\lim_{t\to 0} \frac{1-e^t}{t} = \frac{-e^{-t}}{1} = -1$$

Hence the limit exist.

$$L\left[\frac{1-e^t}{t}\right] = \int_s^{\infty} L(1-e^t)\, ds$$

$$= \int_s^{\infty} \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \left[\log s - \log(s-1)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^{\infty}$$

$$= 0 - \log\left(\frac{s}{s-1}\right)$$

$$= -\log\left(\frac{s}{s-1}\right)$$

$$= \log\left(\frac{s-1}{s}\right) //$$

② Find $L\left[\frac{\sin at}{t}\right]$

Soln:

$$f(t) = \sin at$$

$$\frac{f(t)}{t} = \frac{\sin at}{t}$$

$$\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{\sin at}{t} = \frac{0}{0} \text{ form}$$

Apply L' Hospitals rule.

$$\lim_{t\to 0} \frac{a\cos at}{1} = \frac{a}{1} = a$$

Hence the limit exists

$$\therefore L\left[\frac{\sin at}{t}\right] = \int_s^\infty L(\sin at)\, ds$$

$$\frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \int_s^\infty \left(\frac{a}{s^2+a^2}\right) ds$$

$$= a\int_s^\infty \frac{1}{s^2+a^2}\, ds$$

$$= \cancel{a} \times \frac{1}{\cancel{a}}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\frac{a}{s}\right)$$

$$\cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$\frac{s}{a} = \cot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \frac{s}{a}.$$

③ Find $L\left[\frac{\cos at}{t}\right]$

Soln:

$$f(t) = \cos at$$

$$\frac{f(t)}{t} = \frac{\cos at}{t}$$

$$\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{\cos at}{t} = \frac{1}{0}\ \cancel{\text{form}} = \infty$$

Apply ~~L' Hospital Rule~~.

$$\underset{t\to 0}{Lt} \frac{\cos at}{t} = \frac{1}{0} = \infty$$

The limit does not exists

$\therefore L\left[\frac{\cos at}{t}\right]$ does not exists

(4) Find $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

Soln:

$$f(t) = e^{-at} - e^{-bt}$$

$$\underset{t\to 0}{Lt} \frac{f(t)}{t} = \underset{t\to 0}{Lt} \frac{e^{-at} - e^{-bt}}{t} = \frac{1-1}{0} = \frac{0}{0} \text{ form}$$

Apply L' Hospital Rule.

$$\underset{t\to 0}{Lt} \frac{-a e^{-at} + b e^{-bt}}{1} = \frac{-a+b}{1} = -a+b$$

$\therefore$ The limit exists

$$\therefore L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty L(e^{-at} - e^{-bt})\,ds$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \Big[\log(s+a) - \log(s+b)\Big]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)_s^\infty$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

(5) Find $L\left[\frac{1-\cos at}{t}\right]$

Soln:

$$f(t) = 1 - \cos at$$

$$\frac{f(t)}{t} = \frac{1-\cos at}{t}$$

$$\underset{t\to 0}{Lt} \frac{f(t)}{t} = \underset{t\to 0}{Lt} \frac{1-\cos at}{t} = \frac{1-1}{0} = \frac{0}{0} \text{ form}$$

Apply L' Hospital Rule.

$\therefore \lim_{t \to 0} \dfrac{a \sin at}{1} = \dfrac{0}{1} = 0$

The limit exists

$$\therefore L\left[\frac{1-\cos at}{t}\right] = \int_s^{\infty} L[1-\cos at]\, ds$$

$$= \int_s^{\infty} \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^{\infty}$$

$$= \left[\log s - \log\sqrt{s^2+a^2}\right]_s^{\infty}$$

$$= \log\left[\frac{s}{\sqrt{s^2+a^2}}\right]_s^{\infty}$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\frac{\sqrt{s^2+a^2}}{s}$$

6 Find $L\left[\dfrac{\sin 3t \cos t}{t}\right]$

$f(t) = \sin 3t \cos t$

$\dfrac{f(t)}{t} = \dfrac{\sin 3t \cos t}{t}$

$uv = uv' + vu'$

$\lim_{t \to 0} \dfrac{f(t)}{t} = \lim_{t \to 0} \dfrac{\sin 3t \cos t}{t} = \dfrac{0}{0}$ form

Apply L' Hospital rule.

$\lim_{t \to 0} \dfrac{\sin 3t \sin t + \cos t \; 3\cos 3t}{1} = \dfrac{3}{1} = 3$

$\therefore$ The limit exists

$$L\left[\frac{\sin 3t \cos t}{t}\right] = \int_s^{\infty} L(\sin 3t \cos t)\, ds$$

$$= \int_s^{\infty} L\left[\frac{1}{2}[\sin 4t + \sin 2t]\, ds\right]$$

$$= \frac{1}{2}\int_s^{\infty} L(\sin 4t) + L(\sin 2t)\, ds$$

$$= \frac{1}{2}\int_s^{\infty}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] ds \qquad \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2} - \left(\tan^{-1}\left(\frac{s}{4}\right)+\tan^{-1}\left(\frac{s}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\pi - \tan^{-1}\left(\frac{s}{4}\right) \cancel{+}\ \tan^{-1}\frac{s}{2}\right]$$

(7) Find $L\left[\frac{\sin at}{t}\right]$ Hence show that $\int_0^{\infty}\frac{\sin t}{t}dt = \frac{\pi}{2}$

Soln:

From the above problem (2)

$$L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right)$$

By definition of Laplace transform

$$\int_0^{\infty} e^{-st}\,\frac{\sin at}{t}\,dt = \tan^{-1}\left(\frac{a}{s}\right)$$

put $a=1$ & $s=0$ on both sides

$$\int_0^{\infty}\frac{\sin t}{t}dt = \tan^{-1}\left(\frac{1}{0}\right)$$

$$= \tan^{-1}(\infty)$$

$$= \frac{\pi}{2}$$

(8) Prove that $\int_0^{\infty} e^{-t}\frac{\sin t}{t}dt = \frac{\pi}{4}$

we proved that $L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right)$

$$\int_0^{\infty} e^{-st}\,\frac{\sin at}{t}\,dt = \tan^{-1}\frac{a}{s}$$

put $a=1$ $s=1$

$$\therefore \int_0^{\infty} e^{-t}\,\frac{\sin t}{t}\,dt = \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

9) Find $L\left[\frac{\sin^2 t}{t}\right]$

Soln:-

$$f(t) = \sin^2 t$$

$$\frac{f(t)}{t} = \frac{\sin^2 t}{t}$$

$$\therefore \lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{\sin^2 t}{t} = 0/0 \text{ from}$$

Apply L' Hospital Rule

$$\lim_{t\to 0} \frac{2\sin t\cos t}{1} = \frac{0}{1} = 0$$

$$\therefore L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L[\sin^2 t]\,ds$$

$$= \int_s^\infty L\left[\frac{1-\cos 2t}{2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty [L(1) - L(\cos 2t)]\,ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

10) Find $L\left[\frac{e^{-t}\sin t}{t}\right]$

Soln:-

$$f(t) = e^{-t}\sin t$$

$$\frac{f(t)}{t} = \frac{e^{-t}\sin t}{t}$$

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{e^{-t}\sin t}{t} = \frac{0}{0} = \text{form}$$

Apply L Hospital Rule.

Find $L\left[e^{2t}\int_0^t \frac{\sin 3t}{t}\,dt\right]$

Soln:-

$$L\left[e^{2t}\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{1}{s}L\left[e^{2t}\frac{\sin 3t}{t}\right]$$

$$= \frac{1}{s}L\left[\frac{\sin 3t}{t}\right]_{s\to s-2}$$

$$= \frac{1}{s}\int_s^\infty L(\sin 3t)\,ds_{\,s\to s-2}$$

$$= \frac{1}{s}\int_s^\infty \left(\frac{3}{s^2+3^2}\right) ds_{\,s\to s-2}$$

$$= \frac{3}{s}\left[\frac{1}{3}\tan^{-1}(s/3)\right]_s^\infty{}_{s\to s-2}$$

$$= \frac{1}{s}\left[\tan^{-1}(s/3)\right]_s^\infty{}_{s\to s-2}$$

$$= \frac{1}{s}\left[\tan^{-1}(\infty) - \tan^{-1}(s/3)\right]_{s\to s-2}$$

$$= \frac{1}{s}\left[\pi/2 - \tan^{-1}(s/3)\right]_{s\to s-2}$$

$$= \frac{1}{s}\left[\cot^{-1}(s/3)\right]_{s\to s-2}$$

$$= \frac{1}{s-2}\left[\cot^{-1}\left(\frac{s-2}{3}\right)\right]$$

$$\lim_{t\to 0} \frac{e^{-t}\cos t - \sin t(-e^{-t})}{1} = \lim_{t\to 0} \frac{e^{-t}\cos t + e^{-t}\sin t}{1} = \frac{1}{1} = 1$$

The limit exists.

$$\therefore L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty L[e^{-t}\sin t]\, ds$$

$$= \int_s^\infty L[\sin t]_{s\to s+1}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s^2+1}\right]_{s\to s+1} ds$$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\, ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s+1)$$

$$= \pi/2 - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

(11) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$\frac{f(t)}{t} = \frac{\cos at - \cos bt}{t}$$

$$\lim_{t\to 0} \frac{\cos at - \cos bt}{t} = \frac{1-1}{0} = \frac{0}{0} \text{ form}$$

Apply L Hospital Rule.

$$\lim_{t\to 0} \frac{a\sin at - b\sin bt}{1} = \frac{a-a}{1} = \frac{a-a}{1} = \frac{0}{1} = 0$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\, ds$$

$$= \int_s^\infty L[\cos at] - L[\cos bt]\, ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\tfrac{1}{2}\log(s^2+a^2) - \tfrac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$\left[\log\sqrt{s^2+a^2} - \log\sqrt{s^2+b^2}\right]_s^\infty$$

$$\log\left[\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right]_s^\infty$$

$$= 0 - \log\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}$$

$$= \log\frac{\sqrt{s^2+b^2}}{\sqrt{s^2+a^2}} \;//.$$

Result:-

1. $L[f'(t)] = s\,L[f(t)] - f(0)$
2. $L[f''(t)] = s^2 L[f(t)] - s\,f(0) - f'(0)$

INITIAL VALUE THEOREM

If $L[f(t)] = F(s)$, then $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$

Proof:-

$$L[f'(t)] = s\,L[f(t)] - f(0)$$
$$= s\cdot F(s) - f(0)$$

$$sF(s) - f(0) = L[f'(t)]$$
$$= \int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to\infty}[sF(s) - f(0)] = \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to\infty}[sF(s) - f(0)] = 0 \quad (\because e^{-\infty} = 0)$$

$$\lim_{s\to\infty} sF(s) = f(0) = \lim_{t\to 0} f(t)$$

Hence $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$,

FINAL VALUE THEOREM

If $L[f(t)] = F(s)$ then $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

Proof:-

$$L[f'(t)] = s\,L[f(t)] - f(0)$$
$$= s\,F(s) - f(0)$$

$$sF(s) - f(0) = L[f'(t)]$$

$$sF(s) - f(0) = \int_0^{\infty} e^{-st} f'(t)\,dt$$

$$\lim_{s\to 0}[sF(s) - f(0)] = \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\,dt$$

$$= \int_0^{\infty} f'(t)\,dt$$

is reciprocal of integration.
Diff and Integration is

$$= \int_0^{\infty} d[f(t)]$$

$$= [f(t)]_0^{\infty}$$

$$\lim_{s\to 0} sF(s) - f(0) = f(\infty) - f(0)$$

$$\lim_{s\to 0} sF(s) - f(0) = f(\infty) - f(0)$$

$$\lim_{s\to 0} sF(s) = f(\infty) = \lim_{t\to\infty} f(t)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

1) Verify initial value theorem and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:

Initial value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$L[f(t)] = F(s)$$

$$= L[1 + e^{-t}(\sin t + \cos t)]$$

$$= L[1] + L[e^{-t}\sin t] + L[e^{-t}\cos t]$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

L.H.S

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left(1 + e^{-t}(\sin t + \cos t)\right)$$

$$= 1+1$$

$$= 2 \quad \text{---} (1)$$

R.H.S

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{1+2/s}{1+2/s+2/s^2}\right]$$

$$= 1 + \frac{1+0}{1+0+0} = 1+1 = 2 \quad \text{—} (2)$$

From (1) & (2)

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Final value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

L.H.S

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \left(1+e^{-t}(\sin t + \cos t)\right)$$

$$= 1+0$$

$$= 1 \quad \text{—} (1)$$

R.H.S

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1+0$$

$$= 1 \quad \text{—} (2)$$

From (1) & (2)

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

(2) Verify the initial & final value theorem for $f(t) = 3e^{-2t}$

Soln:-

Initial value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$F(s) = L[f(t)] = L[3e^{-2t}]$$

$$= \frac{3}{s+2}$$

L.H.S $\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{3e^{-2t}}{\cancel{s+2}}$

$= \lim_{t \to 0} 3e^{-2t} = 3$ —①

R.H.S $\lim_{s \to \infty} s F(s) = \lim_{s \to \infty} s\left[\frac{3}{s+2}\right]$

$= \lim_{s \to \infty} \frac{3s}{s+2} = \lim_{s \to \infty} \frac{3}{1+2/s} = \frac{3}{1+0} = 3/1 = 3$ — ②

From (1) & (2)

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$

Final value theorem states that

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$

L.H.S $\lim_{t \to \infty} 3e^{-2t} = 3e^{-\infty} = 3(0) = 0$ — ①

R.H.S $\lim_{s \to 0} \frac{3s}{s+2} = \frac{0}{0+2} = 0$ — ②

From ① & ②

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$

③ If $L[f(t)] = \frac{1}{s(s+a)}$ find $\lim_{t \to \infty} f(t)$ and $\lim_{t \to 0} f(t)$

Soln:-

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s . F(s)$

~~L.H.S~~ $\lim_{t \to 0} f(t) = \lim_{s \to \infty} \cancel{s} \frac{1}{\cancel{s}(s+a)}$

$= \lim_{s \to \infty} \frac{1}{s+a} = \frac{1}{\infty} = 0$

$\lim_{t \to \infty} f(t) = 1/a$

$\lim_{t \to 0} f(t) = 0$

~~R.H.S~~

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s . F(s)$

$= \lim_{s \to 0} \cancel{s} . \frac{1}{\cancel{s}(s+a)}$

$= \lim_{s \to 0} \frac{1}{s+a} = \frac{1}{0+a} = 1/a.$

TRANSFORMS OF PERIODIC FUNCTIONS

Defn:-

The Laplace transform of periodic function of $f(t)$ with period t is given by $\boxed{L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt}$

① Find the Laplace transform of the Half wave rectifier function $f(t) = \begin{cases} \sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

Soln:-

This function is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $(0, \frac{2\pi}{\omega})$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (0)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{s^2+\omega^2}(s\sin\omega t - \omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-s\frac{\pi}{\omega}}}{s^2+\omega^2}(\omega) + [\omega]\right]$$

$$= \frac{1}{1-(e^{-\frac{s\pi}{\omega}})^2}\,\omega\left[\frac{1+e^{-s\frac{\pi}{\omega}}}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(1+e^{-\frac{s\pi}{\omega}}\right)}{\left(1+e^{-\frac{s\pi}{\omega}}\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}$$

$$= \frac{\omega}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)} \;//.$$

25

(5) Find the Laplace transform of $f(t) = \begin{cases} t & 0 < t < a \\ 2a-t & a < t < 2a \end{cases}$ with $f(t+2a) = f(t)$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{a} e^{-st}\, t\, dt + \int_a^{2a} e^{-st}(2a-t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}} \left[a\left(\frac{e^{-sa}}{-s}\right) - \left(\frac{e^{-sa}}{s^2}\right) - \left(-\frac{1}{s^2}\right)\right] + \left[\frac{e^{-2as}}{s^2} - \left(a\frac{e^{-sa}}{s} + \frac{e^{-sa}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-a e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2}\right] + \left[\frac{e^{-2as}}{s^2} - \frac{a e^{-sa}}{s} - \frac{e^{-sa}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-sa}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}} \left[1 - e^{-sa}\right]^2 / s^2$$

$$= \frac{(1-e^{-sa})^{\cancel{2}}}{(1+e^{-sa})\cancel{(1-e^{-sa})}\, s^2}$$

$$= \frac{1-e^{-sa}}{s^2(1+e^{-sa})} = \frac{1}{s^2} \tanh\left[\frac{as}{2}\right] //$$

Note:-

$$\frac{1-e^{-as}}{1+e^{-as}} = \frac{e^{-as/2} e^{as/2} - e^{-as/2} \times e^{-as/2}}{e^{-as/2} e^{as/2} + e^{-as/2} e^{-as/2}}$$

$$= \frac{\cancel{e^{-as/2}}\left[e^{as/2} - e^{-as/2}\right]}{\cancel{e^{-as/2}}\left[e^{as/2} + e^{-as/2}\right]}$$

$$= \tanh\left[\frac{as}{2}\right] //$$

3. Find the Laplace transform of rectangular wave given by

$$f(t) = \begin{cases} 1 & 0<t<b \\ -1 & b<t<2b \end{cases}$$

Soln:-

This function is a periodic function with period 2b in the interval (0, 2b)

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

$\cosh x =$

tan

$$L[f(t)] = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-sb}}{-s} - \frac{1}{-s}\right) - \left(\frac{e^{-2bs}}{-s} - \frac{e^{-bs}}{-s}\right)\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1}{s}\left[-e^{-sb}+1+e^{-2bs}-e^{-bs}\right]\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1}{s}\left[1+e^{-2bs}-2e^{-sb}\right]\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1}{s}\left(1-e^{-bs}\right)^2\right]$$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}\left(1-e^{-bs}\right)^2$$

$$= \frac{1}{s}\,\frac{[1-e^{-bs}]}{[1+e^{-bs}]} = \frac{\frac{1}{s}\left[e^{+bs/2}e^{-bs/2} - e^{-bs/2}e^{-bs/2}\right]}{\left(e^{bs/2}e^{-bs/2} + e^{-bs/2}e^{-bs/2}\right)}$$

$$= \frac{1}{s}\,\frac{e^{-bs/2}}{e^{-bs/2}}\,\frac{\left[e^{bs/2} - e^{-bs/2}\right]}{\left[e^{bs/2} + e^{-bs/2}\right]}$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right)$$

④ Find the Laplace transform of $|\sin t|$

Soln:-

$|\sin t|$ is a periodic function with period $(0, \pi)$

$$L[f(t)] = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} \right]$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{1+e^{-\pi s}}{s^2+1} \right]$$

$$= \frac{1}{s^2+1} \left[\frac{1+e^{-\pi s}}{1-e^{-\pi s}} \right]$$

$$= \frac{1}{s^2+1} \coth\left(\frac{\pi s}{2}\right) //$$

⑤ Find the Laplace transform of $f(t) = \{t, \quad 0 \le t \le 4\}$ if $f(t+4) = f(t)$ for all $t \ge 0$.

Soln:-

$f(t)$ is a periodic function with period 4 the interval is $(0, 4)$

$$\therefore L[f(t)] = \frac{1}{1-e^{-4s}} \int_0^4 t\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^4$$

$$= \frac{1}{1-e^{-4s}} \left[4\left(\frac{e^{-4s}}{-s}\right) - \left(\frac{e^{-4s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right) \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{-4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} \right] //$$

① Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi, \end{cases} \quad f(t+2\pi) = f(t)$$

Ans:- $\dfrac{1}{(1-e^{-s\pi})(1+s^2)}$

② Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} E, & 0 \leq t \leq \frac{1}{E} \\ 0, & \frac{1}{E} \leq t < \frac{2}{E} \end{cases}$$

Ans: $\dfrac{E}{s\left(1+e^{-\frac{s}{E}}\right)}$

③ Find the Laplace Transform of $f(t) = t^2$, $0<t<2$ which is extended periodically with period $t=2$

Ans:- $\dfrac{1}{1-e^{-2s}}\left[-2e^{-2s}\left[\dfrac{1+2s+2s^2}{s^2}\right]+\dfrac{2}{s^3}\right]$

④ Find the L.T. of $f(t) = e^{-t}$ $0 \leq t \leq 2$ $f(t+2) = f(t)$.

Ans:- $\dfrac{1-e^{-2(s+1)}}{(s+1)\left(1-e^{-2s}\right)}$

INVERSE LAPLACE TRANSFORM

Important formulae:-

If $L[f(t)] = \phi(s)$ then

$L^{-1}[\phi(s)] = f(t)$.

Where L^{-1} is called the Inverse Laplace transform operator.

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

4. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

5. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

6. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

7. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

8. $L^{-1}[\phi(s-a)] = e^{at}f(t)$ and $L^{-1}[\phi(s+a)] = e^{-at}f(t)$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

11. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sinh bt$

12. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt$

13. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

14. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$

15. $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

16. $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[\sin at + at\cos at]$

Problems:-

1. Find $L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$

2. Find $L^{-1}\left[\frac{1}{s^2-25}\right] = \frac{1}{5}\sinh 5t$

3. Find $L^{-1}\left[\frac{2s}{s^2-16}\right] = 2L^{-1}\left[\frac{s}{s^2-16}\right] = 2\cosh 4t$

4. Find $L^{-1}\left[\frac{1}{s^2+25}\right] = \frac{1}{5}\sin 5t$

5. Find $L^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{2t}\sin t$

6. Find $L^{-1}\left[\frac{s-1}{(s-1)^2+4}\right] = e^{t}L^{-1}\left[\frac{s}{s^2+4}\right] = e^{t}\cos 2t$

7. Find $L^{-1}\left[\frac{1}{(s+3)^2-4}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2-4}\right] = e^{-3t}\frac{1}{2}\sinh 2t = \frac{1}{2}e^{-3t}\sinh 2t$

8. Find $L^{-1}\left[\frac{s+2}{(s+2)^2-36}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2-36}\right] = e^{-2t}\cosh 6t$

9. Find $L^{-1}\left[\frac{1}{3s-7}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{s-7/3}\right] = \frac{1}{3}e^{7/3\,t}$

10. Find $L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-9}\right] = e^{3t} + 1 + \cosh 3t$

11. Find $L^{-1}\left[\frac{2s^2-4s+5}{s^3}\right]$

$$= L^{-1}\left[\frac{2/s - 4/s^2 + 5/s^3}{1}\right]$$

$$= L^{-1}\left[2/s - 4/s^2 + 5/s^3\right]$$

$$= 2L^{-1}(1/s) - 4L^{-1}(1/s^2) + 5L^{-1}(1/s^3)$$

$$= 2(1) - 4\left(\frac{t}{1!}\right) + 5\left(\frac{t^2}{2!}\right)$$

$$= 2 - 4t + \frac{5t^2}{2}$$

12. Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Soln:

$$L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2}\right] - L^{-1}\left[\frac{2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-2t} - 2e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-2t} - 2e^{-2t}\frac{t}{1!}$$

$$= e^{-2t} - 2e^{-2t}t$$

$$= e^{-2t}[1-2t]$$

13. Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$

$$L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - L^{-1}\left[\frac{2}{(s+2)^2+1}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+1}\right] - 2e^{-2t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos 2t - 2e^{-2t}\sin t$$

$$= e^{-2t}[\cos 2t - 2\sin t]$$

14. Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

Soln:

$$= L^{-1}\left[\frac{s-3}{s^2+4s+4+9}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2-5}{(s+2)^2+9}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+9}\right] - 5e^{-2t} L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}\cos 3t - 5e^{-2t}\frac{\sin 3t}{3}$$

15. Find $L^{-1}\left[\frac{2s+1}{s^2+4s+13}\right]$

$$L^{-1}\left[\frac{2s+1}{s^2+4s+4+9}\right] = L^{-1}\left[\frac{2(s+2)-4+1}{(s+2)^2+9}\right]$$

$$= 2L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - L^{-1}\left[\frac{3}{(s+2)^2+9}\right]$$

$$= 2e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right] - e^{-2t} L^{-1}\left[\frac{3}{s^2+9}\right]$$

$$= 2e^{-2t}\cos 3t - e^{-2t}\sin 3t$$

16. Find $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$

Soln:

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A}{(s^2+a^2)} + \frac{B}{(s^2+b^2)}$$

$$A(s^2+b^2) + B(s^2+a^2) = 1$$

Put $s^2=-b^2$

$$B(a^2-b^2) = 1$$

$$B = \frac{1}{a^2-b^2}$$

Put $s^2=-a^2$

$$A(b^2-a^2) = 1$$

$$A = \frac{1}{b^2-a^2}$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{\frac{1}{b^2-a^2}}{(s^2+a^2)} + \frac{\frac{1}{a^2-b^2}}{(s^2+b^2)}\right]$$

$$= \frac{1}{b^2-a^2} L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

17. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln:

$$L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s^2+a^2}{(s^2+a^2)(s^2+b^2)}\right] - L^{-1}\left[\frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2}\right] - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

18. Find $L^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right]$

$$= L^{-1}\left[\frac{s^2+w^2-w^2}{(s^2+w^2)^2}\right]$$

$$= L^{-1}\left[\frac{s^2+\omega^2}{(s^2+\omega^2)^2}\right] - L^{-1}\left[\frac{\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+\omega^2)}\right] - \omega^2 \cdot L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{\omega}\sin\omega t - \omega^2\left(\frac{1}{2\omega^3}\right)\left[\sin\omega t - \omega t\cos\omega t\right]$$

$$= \frac{1}{\omega}\sin\omega t - \frac{1}{2\omega}\left[\sin\omega t - \omega t\cos\omega t\right]$$

$$= \left[\frac{2\sin\omega t - \sin\omega t + \omega t\cos\omega t}{2\omega}\right]$$

$$= \frac{\sin\omega t + \omega t\cos\omega t}{2\omega} //$$

19. Find $L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right]$

Soln:

Multiply and Divide by $2\omega^2$ also adding & Sub by s^2

$$= L^{-1}\left[\frac{(s^2+\omega^2)-(s^2-\omega^2)}{2\omega^2(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^2}L^{-1}\left[\frac{s^2+\omega^2}{(s^2+\omega^2)^2}\right] - L^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^2}L^{-1}\left[\frac{1}{(s^2+\omega^2)}\right] - t\cos\omega t$$

$$= \frac{1}{2\omega^2}\left[\frac{1}{\omega}\sin\omega t - t\cos\omega t\right]$$

$$= \left[\frac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}\right] //$$

20. Find $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

$$= L^{-1}\left[\frac{2s}{9s^2-25}\right] - L^{-1}\left[\frac{5}{9s^2-25}\right]$$

$$= \frac{2}{9}L^{-1}\left[\frac{s}{s^2-25/9}\right] - \frac{5}{9}L^{-1}\left[\frac{1}{s^2-25/9}\right]$$

$$= \frac{2}{9}\cosh(5/3)t - \frac{5}{9}\,\frac{\sinh 5/3\, t}{5/3}$$

$$= \frac{2}{9}\cosh(5/3)t - \frac{5}{9}\times\frac{3}{5}\sinh(5/3\, t)$$

$$= \frac{2}{9}\cosh(5/3)t - \frac{1}{3}\sinh(5/3\, t) //$$

MULTIPLICATION BY 's'

If $L^{-1}[s\phi(s)] = \frac{d}{dt} f(t) + f(0)\delta(t)$

$\delta(t)$ is always 1, $f(t) = \frac{d}{dt}[L^{-1}[\phi(s)]]$, $L^{-1}\phi(s) = f(t)$ then

$L^{-1}[s\phi(s)] = \frac{d}{dt} f(t) + f(0)\delta(t)$

If $L[f(t)] = \phi(s)$ then

$L[f'(t)] = sL[f(t)] - f(0)$

If $L^{-1}[\phi(s)] = f(t)$

$L^{-1}[s\phi(s)] = f'(t)$

$= \frac{d}{dt} f(t)$

$= \frac{d}{dt} L^{-1}[\phi(s)]$ provided $f(0) = 0$ $L^{-1}[f(0)] = 0$ when $t \to 0$

$\therefore L^{-1}[s\phi(s)] = \frac{d}{dt} f(t) + f(0)\delta(t)$

Problems:-

① Find $L^{-1}\left[\frac{s}{s^2+1}\right]$

Soln:

Let $\phi(s) = \frac{1}{s^2+1}$

$L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = f(t)$

$f(0) = \sin 0 = 0$

$\therefore L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}\sin t + 0$

$= \cos t$ //

② Find $L^{-1}\left[\frac{s}{4s^2-25}\right]$

Let $\phi(s) = \frac{1}{4s^2-25}$

$L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{4(s^2-\frac{25}{4})}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s^2-\frac{25}{4}}\right]$

$= \frac{1}{4}\,\frac{\sinh(\frac{5}{2})t}{\frac{5}{2}} = \frac{1}{4}\times\frac{2}{5}\sinh(\frac{5}{2}t) = \frac{1}{10}\sinh(\frac{5}{2}t) = f(t)$

$f(0) = 0$

$L^{-1}\left[\frac{s}{4s^2-25}\right] = \frac{d}{dt} f(t) + f(0)\delta(t)$

$= \frac{d}{dt}\left[\frac{1}{10}\sinh(\frac{5}{2}t)\right] + 0$

$= \frac{1}{10}\,\frac{5}{2}\cosh(\frac{5}{2}t)$

$= \frac{1}{4}\cosh(\frac{5}{2}t)$ //

③. Find $L^{-1}\left[\frac{3s}{2s+9}\right]$

Soln:-

Let $\phi(s) = \frac{3}{(2s+9)}$

$\therefore L^{-1}(\phi(s)) = L^{-1}\left[\frac{3}{2s+9}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{1}{s+\frac{9}{2}}\right]$

$= \frac{3}{2} e^{-9/2\, t} = f(t)$

$f(0) = \frac{3}{2} e^{-0}$

$f(0) = \frac{3}{2}$

$\therefore L^{-1}\left[\frac{3s}{(2s+9)}\right] = \frac{d}{dt}\left[\frac{3}{2} e^{-9/2\, t}\right] + \frac{3}{2}(1)$

$= \frac{3}{2} \times -\frac{9}{2} e^{-9/2\, t} + \frac{3}{2}$

$= \frac{-27}{4} e^{-9/2\, t} + \frac{3}{2}$ //

4. Find $L^{-1}\left[\frac{s^2}{(s-2)^2}\right]$

Soln:

Let $\phi(s) = \frac{1}{(s-2)^2}$

$L^{-1}(\phi(s)) = L^{-1}\left[\frac{1}{(s-2)^2}\right] = e^{2t} L^{-1}\left[\frac{1}{s^2}\right] = e^{2t}\, t = f(t)$

$f(0) = 0$

$L^{-1}\left[\frac{s}{(s-2)^2}\right] = \frac{d}{dt}(e^{2t} t) + 0$

$= t[2e^{2t}] + e^{2t}(1)$

$= 2t e^{2t} + e^{2t} = f_1(t)$

$f_1(0) = 1$

$L^{-1}\left[\frac{s^2}{(s-2)^2}\right] = \frac{d}{dt}\left[2te^{2t} + e^{2t}\right] + 1$

$= \left[2t(2e^{2t}) + e^{2t}(2) + 2e^{2t}\right] + 1$

$= \left[4te^{2t} + 2e^{2t} + 2e^{2t}\right] + 1$

$= 4te^{2t} + 4e^{2t} + 1$ //

MULTIPLICATION BY '$\frac{1}{s}$'

Formulae:-

$$L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t L^{-1}[\phi(s)]\, dt$$

Problems:

(1) Find $L^{-1}\left[\frac{1}{s(s+3)}\right]$

$$= \int_0^t L^{-1}\left[\frac{1}{s+3}\right] dt$$

$$= \int_0^t e^{-3t}\, dt$$

$$= \left.\frac{e^{-3t}}{-3}\right]_0^t$$

$$= \frac{e^{-3t}}{-3} - \frac{1}{-3}$$

$$= \frac{e^{-3t} - 1}{-3}$$

$$= \frac{1 - e^{-3t}}{3} \;//$$

② $L^{-1}\left[\frac{1}{s^2(s+a)}\right]$

$$= \int_0^t \int_0^t L^{-1}\left[\frac{1}{s+a}\right] dt\, dt$$

$$= \int_0^t \int_0^t e^{-at}\, dt\, dt = \int_0^t \left[\frac{e^{-at}}{-a}\right]_0^t dt$$

$$= \int_0^t \left[\frac{e^{-at}}{-a} - \frac{1}{-a}\right] dt = \int_0^t \frac{1 - e^{-at}}{a}\, dt$$

$$= \frac{1}{a}\int_0^t \left[1 - e^{-at}\right] dt = \frac{1}{a}\left[t + \frac{e^{-at}}{a}\right]_0^t = \frac{1}{a}\left[t + \frac{e^{-at}}{a} - \frac{1}{a}\right] //$$

③ Find $L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2-2s+5}\right] dt$$

$$= \int_0^t L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right] dt = \int_0^t e^t L^{-1}\left[\frac{1}{s^2+2^2}\right] dt$$

$$= \int_0^t e^t \frac{\sin 2t}{2}\, dt = \frac{1}{2}\int_0^t e^t \sin 2t\, dt$$

$$= \frac{1}{2}\left[\frac{e^t}{1+2^2}\left(\sin 2t - 2\cos 2t\right)\right]_0^t$$

$$= \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t)\right]_0^t = \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t) - 1(0-2)\right]$$

$$= \frac{1}{10}\left[e^{t}(\sin 2t - 2\cos 2t) + 2\right]$$

④ Find $L^{-1}\left[\frac{s^2+3}{s(s^2+9)}\right]$

$$L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t L^{-1}[\phi(s)]\,dt$$

$$L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right] = L^{-1}\left[\frac{s^2+9}{s(s^2+9)}\right] - L^{-1}\left[\frac{6}{s(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{6}{s(s^2+9)}\right]$$

$$= 1 - 6\int_0^t L^{-1}\left[\frac{1}{(s^2+9)}\right]dt$$

$$= 1 - 6\int_0^t \frac{\sin 3t}{3}\,dt$$

$$= 1 - 2\left[-\frac{\cos 3t}{3}\right]_0^t = 1 + \tfrac{2}{3}\left[\cos 3t\right]_0^t$$

$$= 1 + \tfrac{2}{3}\left[\cos 3t - 1\right]$$

⑤ Find $L^{-1}\left[\frac{1}{s(s+2)^3}\right]$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right]dt$$

$$= \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^3}\right]dt$$

$$= \int_0^t e^{-2t}\,\frac{t^2}{2}\,dt$$

$$= \tfrac{1}{2}\left[t^2\left(\frac{e^{-2t}}{-2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$= \tfrac{1}{2}\left\{\left[\frac{-t^2 e^{-2t}}{2} - \frac{t e^{-2t}}{2} - \frac{e^{-2t}}{4}\right] - \left(0 + 0 - \tfrac{1}{4}\right)\right]$$

$$= \tfrac{1}{8}\left[1 - 2t^2 e^{-2t} - 2te^{-2t} - e^{-2t}\right]$$

Inverse Laplace transform of Derivatives

Formula:-

If $L^{-1}[\phi(s)] = f(t)$, then $L^{-1}[\phi'(s)] = -t\, L^{-1}[\phi(s)]$

Problems:-

① Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

Soln:-

let $\phi'(s) = \frac{s+2}{(s^2+4s+5)^2}$

$\int \phi'(s)\,ds = \int \frac{(s+2)}{(s^2+4s+5)^2}\,ds$

$t = s^2+4s+5$

$\frac{dt}{ds} = 2s+4$

$dt = 2(s+2)\,ds$

$\phi(s) = \int \frac{dt/2}{t^2}$

$= \frac{1}{2}\int t^{-2}\,dt$

$= \frac{1}{2}\left[\frac{t^{-1}}{-1}\right]$

$= -\frac{1}{2t} = -\frac{1}{2(s^2+4s+5)}$

$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = -t\, L^{-1}\left[\frac{-1}{2(s^2+4s+5)}\right]$

$= \frac{t}{2}\, L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= \frac{t}{2}\, e^{-2t} L^{-1}\left[\frac{1}{s^2+1}\right]$

$= \frac{t}{2}\, e^{-2t} \sin t$ //

② Find $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$

Soln:

Let $\phi'(s) = \frac{s}{(s^2-a^2)^2}$

$t = s^2-a^2$

$dt = 2s\,ds$

$\frac{dt}{2} = s\,ds$

$\int \phi'(s)\,ds = \int \frac{s}{(s^2-a^2)^2}\,ds$

$\phi(s) = \int \frac{dt/2}{t^2}$

$= \frac{1}{2}\int t^{-2}\,dt$

$= \frac{1}{2}\left[\frac{t^{-1}}{-1}\right]$

$= -\frac{1}{2t} = \frac{-1}{2(s^2-a^2)}$

$\therefore L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = -t\ L^{-1}\left[\frac{-1}{2(s^2-a^2)}\right]$

$= \frac{t}{2}\ L^{-1}\left[\frac{1}{s^2-a^2}\right]$

$= \frac{t}{2}\frac{1}{a}\sinh at$

$= \frac{t}{2a}\sinh at$ //

(3) Find $L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right]$

Soln:

Let $\phi'(s) = \frac{2(s+1)}{(s^2+2s+2)^2}$ $\qquad t = s^2+2s+2$

$\int \phi'(s)\,ds = \int \frac{2(s+1)}{(s^2+2s+2)^2}\,ds$ $\qquad \frac{dt}{ds} = 2s+2$

$\qquad dt = 2(s+1)\,ds$

$= \int \frac{dt}{t^2}$

$= \int t^{-2}\,dt = \frac{t^{-1}}{-1} = \frac{-1}{t}$

$= \frac{-1}{(s^2+2s+2)}$

$L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right] = -t\ L^{-1}\left[\frac{1}{-(s^2+2s+2)}\right]$

$= t\ L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$= t\ e^{-t}\ L^{-1}\left[\frac{1}{s^2+1}\right]$

$= t\ e^{-t}\sin t$ //

(4) Find $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

Soln:

$L^{-1}[\phi'(s)] = -t\ L^{-1}[\phi(s)]$

$L^{-1}[\phi(s)] = \frac{-1}{t}\ L^{-1}[\phi'(s)]$

Let $\phi(s) = \tan^{-1}(1/s)$

$$L^{-1}\phi(s) = L^{-1}\left[\tan^{-1}(1/s)\right]$$ $\tan^{-1}x = \frac{1}{1+x^2}$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}(1/s)\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+1/s^2}\left(-1/s^2\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{s^2}{s^2+1}\left(-1/s^2\right)\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 1/t \; L^{-1}\left[\frac{1}{s^2+1}\right] = 1/t \sin t$$

⑮ Find $L^{-1}\left[\cot^{-1}(1+s)\right]$

Soln:

$$L^{-1}\phi(s) = L^{-1}\left[\cot^{-1}(1+s)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}(1+s)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{1+(1+s)^2}(1)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = 1/t \; e^{-t}\sin t$$

⑯ Find $L^{-1}\left[\tan^{-1}(s)\right]$

Soln:

Let $\phi(s) = \tan^{-1}(s)$

$$L^{-1}\phi(s) = L^{-1}\left[\tan^{-1}(s)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\tan^{-1}(s)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= -\frac{1}{t}\sin t \,//$$

⑰ Find $L^{-1}\left[\tan^{-1}(a/s) + \cot^{-1}(s/b)\right]$

Soln:

Let $\phi(s) = \tan^{-1}(a/s) + \cot^{-1}(s/b)$

$$L^{-1}\phi(s) = L^{-1}\left[\tan^{-1}(a/s) + \cot^{-1}(s/b)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}(a/s) + \cot^{-1}(s/b)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+a^2/s^2}\left(\frac{-a}{s^2}\right) - \frac{1}{1+s^2/b^2}\left(\frac{1}{b}\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{a}{s^2+a^2} + \frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{t}\left[\sin at + \sin bt\right]$$

(8) Find $L^{-1}\left[\log\left(\frac{s^2-1}{s^2}\right)\right]$

Soln:

$$L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$$

$$L^{-1}\left[\log\left(\frac{s^2-1}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2-1}{s^2}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\log(s^2-1) - \log s^2\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2-1} - 2\log s\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2-1} - 2\left(\frac{1}{s}\right)\right]$$

$$= \frac{-1}{t}\left[2\cosh t - 2(1)\right]$$

$$= \frac{-2}{t}\left[\cosh t - 1\right]$$

$$= \frac{2}{t}\left[1-\cosh t\right]$$

(9) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Soln:

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(s+1) - \log(s-1)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}\left[e^{-t} - e^{t}\right]$$

$$= \frac{1}{t}\left[e^{t} - e^{-t}\right]$$

$$= \frac{1}{t}\left[2\sinh t\right]$$

(10) Find $L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right]$

Soln:

$$L^{-1}\left[\log\left(\frac{s^2+\omega^2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+\omega^2}{s^2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s^2+\omega^2) - \log s^2\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right]$$

$$= -\frac{1}{t}\left[2\cos\omega t - 2(1)\right]$$

$$= \frac{-2}{t}\left[\cos\omega t - 1\right]$$

$$= \frac{2}{t}\left[1-\cos\omega t\right]$$

11. Find $L^{-1}\left[\log\left(\frac{s(s+1)}{(s^2+1)}\right)\right]$

Soln:

$$L^{-1}\left[\log\frac{s(s+1)}{(s^2+1)}\right] = -\frac{1}{t}L^{-1}\frac{d}{ds}\left[\log\frac{s(s+1)}{s^2+1}\right]$$

$$= -\frac{1}{t}L^{-1}\frac{d}{ds}\left[\log s(s+1) - \log(s^2+1)\right]$$

$$= -\frac{1}{t}L^{-1}\frac{d}{ds}\left[\log s + \log(s+1) - \log(s^2+1)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}\right]$$

$$= -\frac{1}{t}\left[1 + e^{-t} - 2\cos t\right]$$

PARTIAL FRACTION METHOD

1. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Soln:

Let $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

put $s=0$, $1 = 2A$, $\boxed{A = 1/2}$ | $s=-1$, $1 = -B$, $\boxed{B=-1}$ | $s=-2$, $1 = 2C$, $\boxed{C = 1/2}$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \left[\frac{1/2}{s} - \frac{1}{(s+1)} + \frac{1/2}{(s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{(s+1)} + \frac{1}{2}\left(\frac{1}{s+2}\right)\right]$$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$= \frac{1 - 2e^{-t} + e^{-2t}}{2}$$ //.

(2) Find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+3)}\right]$

Soln:

let $\frac{1-s}{(s+1)[(s+1)(s+3)]} = \frac{1-s}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$

$$1-s = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

put $s=-1$, $2 = 2B$, $B = 2/2$, $\boxed{B=1}$ | Put $s=-3$, $4 = 4C$, $C = 4/4$, $\boxed{C=1}$ | Equating constant term: $1 = 3A + 3B + C$, $1 = 3A + 3 + 1$, $1 - 4 = 3A$, $-3 = 3A$, $A = -3/3 = -1$, $\boxed{A=-1}$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+3)}\right] = L^{-1}\left[\frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+3}\right]$$

$$= -e^{-t} + e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + e^{-3t}$$

$$= -e^{-t} + e^{-t}t + e^{-3t}$$

③ Find $L^{-1}\left[\dfrac{1-s}{(s+1)(s^2+4s+13)}\right]$

let $\dfrac{1-s}{(s+1)(s^2+4s+13)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+4s+13}$

$$1-s = A(s^2+4s+13) + Bs+C(s+1)$$

put $s=-1$

$2 = A(1-4+13)$

$2 = A(10)$

$A = 2/10$

$\boxed{A = 1/5}$

Equating the s^2 & constant terms

$A+B=0$

$1/5 + B = 0$

$\boxed{B = -1/5}$

$1 = 13A + C$

$1 = 13(1/5) + C$

$1 - \frac{13}{5} = C$

$\boxed{C = -8/5}$

$$\therefore L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = L^{-1}\left[\frac{1/5}{s+1} + \frac{-\frac{1}{5}s - 8/5}{s^2+4s+13}\right]$$

$$= \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+8}{(s+2)^2+9}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] + \frac{-6}{5}L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right] - \frac{6}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$$

④ Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

⑤ Find $L^{-1}\left[\frac{3(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right]$ (17)

Soln

Let

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s-2)(s+1) + D(s+1)$$

put $s=-1$

$5+15-11 = A(-3)^3$

$20-11 = A(-27)$

$9 = -27A$

$A = -\frac{9}{27}$

$\boxed{A = -\frac{1}{3}}$

put $s=2$

$5(4)-15(2)-11 = D(3)$

$20-30-11 = 3D$

$20-41 = 3D$

$\frac{-21}{3} = D$

$\boxed{D=-7}$

Equating the coefficient of s^3

$0 = A+B$

$A+B=0$

$-\frac{1}{3}+B=0$

$\boxed{B=\frac{1}{3}}$

put $s=0$ we get

$-11 = -8A+4B-2C+D$

$-11 = -8\left(-\frac{1}{3}\right)+4\left(\frac{1}{3}\right)-2C+(-7)$

$-11 = \frac{8}{3}+\frac{4}{3}-2C-7$

$-11 = \frac{8+4-6C-21}{3}$

$-33 = 12-21-6C$

$-33 = -9-6C$

$-33+9 = -6C$

$-24 = -6C \quad = \quad C = \frac{-24}{-6}$

$\boxed{C=4}$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = L^{-1}\left[\frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}\right]$$

$$= -\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4e^{+2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t} + -7e^{2t}\frac{t^2}{2!}$$

CONVOLUTION

Defn:-

The convolution of two functions $f(t)$ and $g(t)$ is defined as $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

State and prove convolution theorem

If $f(t)$ and $g(t)$ are function defined for $t \geq 0$, then

$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$

Proof:

By definition

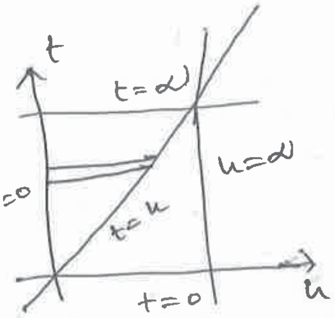
$$L[f(t)] = \int_0^\alpha e^{-st} f(t)\, dt$$

$$\therefore L[f(t)*g(t)] = \int_0^{\infty} e^{-st}[f(t)*g(t)]\,dt$$

$$= \int_0^{\infty} e^{-st}\left[\int_0^t f(u)\,g(t-u)\,du\right] dt \qquad \text{(By defn)}$$

$$= \int_{t=0}^{t=\infty}\int_{u=0}^{u=t} e^{-st} f(u)\,g(t-u)\,du\,dt$$

By changing the order of integration



Given $t=0$ to $t=\infty$

$u=0$ to $u=t$

$$= \int_0^{\infty}\int_u^{\infty} e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^{\infty} f(u)\int_u^{\infty} e^{-st}\,g(t-u)\,dt\,du$$

put $t-u=v$ | $t\to u \Rightarrow v\to 0$

$dt = dv$ | $t\to\infty \Rightarrow v\to\infty$

$$\therefore \int_0^{\infty} f(u)\int_0^{\infty} e^{-(u+v)s}\,g(v)\,dv\,du$$

$$= \int_0^{\infty} f(u)\,e^{-us}\,du\int_0^{\infty} e^{-vs}\,g(v)\,dv$$

$$= \int_0^{\infty} e^{-us} f(u)\,du\int_0^{\infty} e^{-vs}\,g(v)\,dv$$

[Since u and v are dummy variable]

$$= \int_0^{\infty} e^{-st} f(t)\,dt\int_0^{\infty} e^{-st} g(t)\,dt$$

$$= L[f(t)]\cdot L[g(t)]$$

Corrollary:-

$$L[f(t)*g(t)] = L[f(t)]\cdot L[g(t)]$$

$$= F(s)\cdot G(s)$$

where $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$

$$L^{-1}[F(s)G(s)] = f(t)*g(t) = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems:-

1. Using convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt} = \int_0^t e^{-au} e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} e^{-bt} e^{bu} du = e^{-bt}\int_0^t e^{-au} e^{bu} du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u} du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right]$$

$$= \frac{e^{-bt}}{(a-b)}\left[1 - e^{-(a-b)t}\right]$$

$$= \frac{e^{-bt}}{(a-b)}\left[1 - e^{-at} e^{+bt}\right]$$

$$= \frac{1}{a-b}\left[e^{-bt} - e^{-at}\right]$$

2. Using convolution theorem find $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

Soln:-

$$L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 1 * \sin t$$

$$= \int_0^t \sin(t-u)\, du$$

$$= \left[\frac{-\cos(t-u)}{-1}\right]_0^t = \cos(t-t) - \cos(t-0)$$

$$= \cos 0 - \cos t$$

$$= 1 - \cos t$$

3. Using convolution theorem, $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Soln:

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{\sin at}{a} = \int_0^t \cos au \, \frac{\sin a(t-u)}{a} du$$

$$\frac{1}{a}\int_0^t \frac{\sin[at-au+au] + \sin[at-au-au]}{2}\,du$$

$\frac{1}{2}\sin(A+B)+\sin(A-B)$

$$\frac{1}{2a}\int_0^t \sin at + \sin(at-2au)\,du$$

$$\frac{1}{2a}\left[(\sin at)\,u + \left(\frac{-\cos(at-2au)}{-2a}\right)\right]_0^t$$

$$\frac{1}{2a}\left[(\sin at)t + \frac{\cos at}{2a} - \left(0+\frac{\cos at}{2a}\right)\right]$$

$$\frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$\frac{1}{2a}\, t\sin at$$ //

① Solve the I. eqn

$y(t) = 1 + \int_0^t y(u)\sin(t-u)\,du$

② Solve the I. eqn

$F'(t) = t + \int_0^t F(t-u)\cos u\,du$

$F(0) = 4.$

④ Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ by using Convolution theorem

Soln:-

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$\cos at * \cos bt = \int_0^t \cos au\,\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)]\,du$$

$$= \frac{1}{2}\int_0^t \cos((a-b)u+bt) + \cos((a+b)u-bt)]\,du$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)u+bt}{a-b} + \frac{\sin(a+b)u-bt}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right] - \left[\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

[Cross multiply this one]

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \left[\frac{a\sin at - b\sin bt}{a^2-b^2}\right]$$ //

(5) Find the $L^{-1}\left[\frac{1}{(s+1)(s^2+9)}\right]$ using convolution theorem

Soln:-

$$L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-t} * \frac{\sin 3t}{3} = \int_0^t e^{-u}\frac{\sin 3(t-u)}{3}\,du$$

$$= \frac{1}{3}\int_0^t e^{-u}[\sin(3t-3u)du]$$

$$= \frac{1}{3}\int_0^t e^{-u}[\sin 3t\cos 3u - \cos 3t \sin 3u]\,du$$

$$= \frac{1}{3}\sin 3t\int_0^t e^{-u}\cos 3u\,du - \frac{1}{3}\cos 3t\int_0^t e^{-u}\sin 3u\,du$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3u + 3\sin 3u)\right]_0^t - \frac{\cos 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3u - 3\cos 3u)\right]_0^t$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-t}}{10}(-\cos 3t + 3\sin 3t) - \frac{1}{10}(-1)\right] - \frac{\cos 3t}{3}\left[\frac{e^{-t}}{10}(-\sin 3t - 3\cos 3t) - \frac{1}{10}(-3)\right]$$

$$= \frac{e^{-t}}{30}\,3(\sin^2 3t + \cos^2 3t) + \frac{1}{30}(\sin 3t - 3\cos 3t).$$

$$= \frac{e^{-t}}{\cancel{30}}\,\cancel{3}(1) + \frac{1}{30}(\sin 3t - 3\cos 3t)$$

$$= \frac{e^{-t}}{10} + \frac{1}{30}(\sin 3t - 3\cos 3t)$$

Application to solution of linear ordinary differential equation upto 2nd order with constant co-efficients

Type (1)

(1) Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$ where $y = 2$, $\frac{dy}{dt} = -4$ at $t = 0$

Soln:-

Given $y''(t) + 2y'(t) + 5y(t) = 0$

Taking Laplace transform on both sides

$$L[y''(t)] + 2L[y'(t)] + 5L[y(t)] = 0$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] + 5L[y(t)] = 0$$

Given

$y(0) = 2$, $y'(0) = -4$

$$[s^2L[y(t)] - s(2) - (-4)] + 2[sL[y(t)] - 2] + 5L[y(t)] = 0$$

$$s^2L[y(t)] - 2s + 4 + 2sL[y(t)] - 4 + 5L[y(t)] = 0$$

$$L[y(t)][s^2 + 2s + 5] - 2s = 0$$

$$L[y(t)][s^2 + 2s + 5] = 2s$$

$$L\,y(t) = \frac{2s}{s^2+2s+5}$$

$$y(t) = L^{-1}\left[\frac{2s}{s^2+2s+5}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2+2s+5}\right]$$

$$= 2L^{-1}\left[\frac{s+1-1}{(s+1)^2+4}\right]$$

$$= 2L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] - 2L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= 2e^{-t}L^{-1}\left[\frac{s}{s^2+4}\right] - 2e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 2e^{-t}\cos 2t - 2e^{-t}\frac{\sin 2t}{2}$$

Type ①

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

② Solve $y'' + 2y' + y = te^{-t}$ if $y(0)=1$ $y'(0)=-2$.

Soln:-

$$y''(t) + 2y'(t) + y(t) = te^{-t}$$

Taking L.T. on both sides

$$L[y''(t)] + 2L[y'(t)] + L[y(t)] = L[te^{-t}]$$

$$\left\{s^2L[y(t)] - sy(0) - y'(0)\right\} + 2\left\{sL[y(t)] - y(0)\right\} + L[y(t)] = -\frac{d}{ds}\left[\frac{1}{s+1}\right]$$

Given $y(0)=1$ $y'(0)=-2$

$$\{s^2L[y(t)] - s + 2\} + \{2sL[y(t)] - 2\} + L[y(t)] = \frac{+1}{(s+1)^2}$$

$$L[y(t)]\,[s^2+2s+1] - s = \frac{1}{(s+1)^2}$$

$$L[y(t)]\,(s+1)^2 = s + \frac{1}{(s+1)^2}$$

$$L[y(t)] = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$y(t) = L^{-1}\left[\frac{s}{(s+1)^2} + \frac{1}{(s+1)^4}\right]$$

$$y(t) = L^{-1}\left[\frac{s+1-1}{(s+1)^2} + \frac{1}{(s+1)^4}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{(s+1)^4}\right]$$

$$= e^{-t} - e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + e^{-t} L^{-1}\left[\frac{1}{s^4}\right]$$

$$= e^{-t} - e^{-t} t + e^{-t} \frac{t^3}{3!}$$

$$= e^{-t} - e^{-t} t + e^{-t} \frac{t^3}{6} = e^{-t}\left(1 - t + \frac{t^3}{6}\right)$$

Type (2)

(3) Using Laplace Transform solve

$$y'' - 3y' + 2y = e^{-t} \quad \text{Given } y(0) = 1,\ y'(0) = 0$$

soln:-

Given $y''(t) - 3y'(t) + 2y(t) = e^{-t}$

Taking L.T. on both sides

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$[s^2 L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

Given $y(0) = 1$ $\quad y'(0) = 0$

$$[s^2 L[y(t)] - s(1) - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$L[y(t)][s^2 - 3s + 2] - s + 3 = \frac{1}{s+1}$$

$$L[y(t)][s^2 - 3s + 2] = \frac{1}{s+1} + s - 3$$

$$= \frac{1 + (s-3)(s+1)}{(s+1)} = \frac{1 + s^2 + s - 3s - 3}{(s+1)} = \frac{s^2 - 2s - 2}{(s+1)}$$

$$L[y(t)] = \frac{s^2 - 2s - 2}{(s+1)[s^2 - 3s + 2]}$$

$$= \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$$

$$y(t) = L^{-1}\left[\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}\right]$$

Consider

$$\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$s^2-2s-2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$

put $s=1$

$1-4 = B(2)(-1)$

$-3 = -2B$

$\boxed{B = 3/2}$

put $s=-1$

$1 = A(-2)(-3)$

$1 = 6A$

$\boxed{A = 1/6}$

put $s=2$

$-2 = C(3)(1)$

$-2 = 3C$

$\boxed{C = -2/3}$

$$\therefore y(t) = L^{-1}\left[\frac{1/6}{s+1} + \frac{3/2}{s-1} - \frac{2/3}{s-2}\right]$$

$$= \frac{1}{6} L^{-1}\left[\frac{1}{s+1}\right] + 3/2\, L^{-1}\left[\frac{1}{s-1}\right] - 2/3\, L^{-1}\left[\frac{1}{s-2}\right]$$

$$= 1/6\, e^{-t} + 3/2\, e^{t} - 2/3\, e^{2t}$$

Type①

④ Solve $y''+4y = \sin 2t$ Given $y(0) = y'(0) = 0$

Soln:-

$$y''(t) + 4y(t) = \sin 2t$$

Taking L.T on both sides

$$L[y''(t)] + 4L[y(t)] = L[\sin 2t]$$

$$[s^2 L[y(t)] - sy(0) - y'(0)] + 4L[y(t)] = \frac{2}{s^2+4}$$

Given $y(0)=0$, $y'(0)=0$

$$s^2 L[y(t)] + 4L[y(t)] = \frac{2}{s^2+4}$$

$$L[y(t)]\,(s^2+4) = \frac{2}{s^2+4}$$

$$L[y(t)] = \frac{2}{(s^2+4)^2}$$

$$y(t) = L^{-1}\left[\frac{2}{(s^2+4)^2}\right]$$

multiply and divide by 4 $= 1/4\left[\frac{(s^2+2^2)-(s^2-2^2)}{(s^2+4)^2}\right]$

$$= 1/4\left[\frac{1}{s^2+2^2} - \frac{s^2-2^2}{(s^2+2^2)^2}\right]$$

$$y(t) = 1/4\, L^{-1}\left[\frac{1}{s^2+2^2}\right] - 1/4\, L^{-1}\left[\frac{s^2-2^2}{(s^2+2^2)^2}\right]$$

$$= 1/4\left[\frac{\sin 2t}{2}\right] - 1/4\left[t \cos 2t\right]$$

$$= \frac{1}{8}\sin 2t - \frac{t}{4}\cos 2t$$

5 Type(I) Solve $\frac{dy}{dt} - y = e^t$ where $y(0) = 1$ by L.T. method

Soln:

$$y'(t) - y(t) = e^t$$

Apply L.T. on both sides

$$L[y'(t)] - L[y(t)] = L[e^t]$$

$$sL[y(t)] - y(0) - L[y(t)] = \frac{1}{s-1}$$

$$sL[y(t)] - L[y(t)] - 1 = \frac{1}{s-1}$$

$$L[y(t)][s-1] = \frac{1}{s-1} + 1$$

$$L[y(t)](s-1) = \frac{1+s-1}{s-1} = \frac{s}{s-1}$$

$$L[y(t)] = \frac{s}{(s-1)^2}$$

$$y(t) = L^{-1}\left[\frac{s}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{s+1-1}{(s-1)^2}\right] = L^{-1}\left[\frac{s-1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] + e^t L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t + e^t \cdot t$$

$$= e^t(1+t)$$

Type(I)

6 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ given that $y = \frac{dy}{dx} = 1$ at $x = 0$

Soln:

Given $y''(x) - 2y'(x) + 2y(x) = 0$ $\quad y(0) = 1, \; y'(0) = 1$

$$L[y''(x)] - 2L[y'(x)] + 2L[y(x)] = 0$$

$$[s^2 L[y(x)] - sy(0) - y'(0)] - 2[sL(y(x)) - y(0)] + 2L[y(x)] = 0$$

$$[s^2 L[y(x)] - s(1) - 1] - 2[sL[y(x)] - 1] + 2L[y(x)] = 0$$

$$s^2 L[y(x)] - s - 1 - 2sL[y(x)] + 2 + 2L[y(x)] = 0$$

$$s^2 L[y(x)] - 2sL[y(x)] + 2L[y(x)] - s + 1 = 0$$

$$L[y(x)][s^2 - 2s + 2] = s - 1$$

$$L[y(x)] = \frac{s-1}{s^2-2s+2}$$

$$= \frac{s-1}{(s-1)^2+1} \quad = L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right]$$

$$y(x) = L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right]$$

$$= e^{x} L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$\boxed{y(x) = e^{x}\cos x}$$

Type 2

(7) Solve by using L.T $(D^2+9)y = \cos 2t$ Given that $y(0)=1$ $y(\pi/2)=-1$

Soln:

Given $(D^2+9)y = \cos 2t$

$$y''(t) + 9y(t) = \cos 2t$$

$$L[y''(t)] + 9L[y(t)] = L[\cos 2t]$$

$$[s^2 L[y(t)] - s(y(0) - y'(0)] + 9L[y(t)] = \frac{s}{s^2+4}$$

Given $y(0)=1$ taking $y'(0)=k$

$$s^2 L[y(t)] - s(1) - k + 9L[y(t)] = \frac{s}{s^2+4}$$

$$s^2 L[y(t)] + 9L[y(t)] - s - k = \frac{s}{s^2+4}$$

$$L[y(t)][s^2+9] = \frac{s}{s^2+4} + s + k$$

$$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s+k}{s^2+9} \quad \text{---} (1)$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{CB+D}{s^2+9} \quad \text{---} (A)$$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

Equating s^3 on bothsides we get $0 = A + C$ ---(2)

Equating s^2 on both sides we get $0 = B + D$ ---(3)

Equating s on both sides we get $1 = 9A + 4C$ ---(4)

Put $s=0$ on both sides we get $0 = 9B + 4D$ ---(5)

$(2) \Rightarrow \quad C = -A$

$\therefore (4) \Rightarrow \quad 9A + 4C = 1$

$9A + 4(-A) = 1$

$9A - 4A = 1$

$5A = 1$

$\boxed{A = 1/5} \qquad \boxed{C = -1/5}$

$(3) \quad D = -B$

$\therefore (5) \quad 9B + 4D = 0$

$9B + 4(-B) = 0$

$9B - 4B = 0$

$5B = 0$

$\boxed{B = 0} \qquad \boxed{D = 0}$

$$(A) \Rightarrow \frac{s}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}s + 0}{s^2+4} + \frac{-\frac{1}{5}s + 0}{s^2+9} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$L[y(t)] = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$y(t) = \frac{1}{5} L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{5} L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + k L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{k}{3}\sin 3t$$

put $t = \pi/2$ we get $y(\pi/2) = \frac{1}{5}\cos 2(\pi/2) - \frac{1}{5}\cos 3(\pi/2) + \cos 3(\pi/2) + \frac{k}{3}\sin 3(\pi/2)$

$$= \frac{1}{5}\cos(\pi) - \frac{1}{5}(0) + 0 + \frac{k}{3}(-1)$$

$$-1 = \frac{1}{5}(-1) - \frac{k}{3} = -\frac{1}{5} - \frac{k}{3}$$

$$-1 = -\frac{1}{5} - \frac{k}{3}$$

$$1 - \frac{1}{5} = \frac{k}{3}$$

$$\frac{4}{5} = \frac{k}{3} \Rightarrow \boxed{k = \frac{12}{5}}$$

$$\therefore y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{\cancel{12}}{\cancel{3} \times 5}\sin 3t$$

$$= \cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$y(t) = \frac{4}{5}[\cos 3t + \sin 3t] + \frac{1}{5}\cos 2t \;//$$

SIMULTANEOUS DIFFERENTIAL EQUATIONS

① Solve $\frac{dx}{dt} - 2x + 3y = 0$, $\frac{dy}{dt} - y + 2x = 0$ with $x(0) = 8$, $y(0) = 3$

Soln:

Given

$$x'(t) - 2x(t) + 3y(t) = 0$$
$$y'(t) - y(t) + 2x(t) = 0$$

Take L.T. we get

$$L[x'(t)] - 2L[x(t)] + 3L[y(t)] = 0$$
$$L[y'(t)] - L[y(t)] + 2L[x(t)] = 0$$
$$sL[x(t)] - x(0) - 2L[x(t)] + 3L[y(t)] = 0$$
$$sL[y(t)] - y(0) - L[y(t)] + 2L[x(t)] = 0$$

Given $x(0) = 8$, $y(0) = 3$

$$sL[x(t)] - 8 - 2L[x(t)] + 3L[y(t)] = 0$$
$$sL[y(t)] - 3 - L[y(t)] + 2L[x(t)] = 0$$

$$(s-2)L[x(t)] + 3L[y(t)] = 8 \quad \text{--- ①}$$
$$(s-1)L[y(t)] + 2L[x(t)] = 3 \quad \text{--- ②}$$

$$L[x(t)] = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{(s+1)(s-4)}$$

$$L[y(t)] = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{(s+1)(s-4)}$$

$$x(t) = L^{-1}\left[\frac{8s-17}{(s+1)(s-4)}\right]$$

$$\frac{8s-17}{(s+1)(s-4)} = \frac{A}{s+1} + \frac{B}{s-4}$$

$$8s-17 = A(s-4) + B(s+1)$$

put $s = -1$

$-25 = A(-5)$

$\boxed{A = 5}$

put $s = 4$

$32 - 17 = B(5)$

$15 = 5B$

$\boxed{B = 3}$

$$y(t) = L^{-1}\left[\frac{3s-22}{(s+1)(s-4)}\right]$$

$$\frac{3s-22}{(s+1)(s-4)} = \frac{A}{s+1} + \frac{B}{s-4}$$

$$3s-22 = A(s-4) + B(s+1)$$

put $s = -1$

$-25 = A(-5)$

$\boxed{A = 5}$

put $s = 4$

$12 - 22 = 5B$

$-10 = 5B$

$\boxed{B = -2}$

$$x(t) = L^{-1}\left[\frac{5}{s+1}\right] + L^{-1}\left[\frac{3}{s-4}\right]$$

$$x(t) = 5e^{-t} + 3e^{4t}$$

$$y(t) = L^{-1}\left[\frac{5}{s+1}\right] - L^{-1}\left[\frac{2}{s-4}\right]$$

$$y(t) = 5e^{-t} - 2e^{4t}$$

Hence $x(t) = 5e^{-t} + 3e^{4t}$

$y(t) = 5e^{-t} - 2e^{4t}$ //

Note: If the simultaneous equations are

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

(2). Solve $\dot{x} + y = \sin t$; $x + \dot{y} = \cos t$ with $x = 2$ and $y = 0$ when $t = 0$

Soln:

Given $x'(t) + y(t) = \sin t$

$x(t) + y'(t) = \cos t$

Apply L.T. on both sides

$$L[x'(t)] + L[y(t)] = L[\sin t]$$

$$L[x(t)] + L[y'(t)] = L[\cos t]$$

$$sL[x(t)] - x(0) + L[y(t)] = \frac{1}{s^2+1}$$

$$L[x(t)] + sL[y(t)] - y(0) = \frac{s}{s^2+1}$$

$$sL[x(t)] + L[y(t)] = 2 + \frac{1}{s^2+1} \quad \text{--- (1)}$$

$$L[x(t)] + sL[y(t)] = \frac{s}{s^2+1} \quad \text{--- (2)}$$

Solving (1) and (2) we get

$$s\,L[x(t)] + L[y(t)] = \frac{2s^2+3}{s^2+1}$$

$$(2)\times s \Rightarrow s\,L[x(t)] + s^2 L[y(t)] = \frac{s^2}{s^2+1}$$

(1) − (2)

$$(1-s^2)\,L[y(t)] = \frac{2s^2+3}{s^2+1} - \frac{s^2}{s^2+1}$$

$$= \frac{2s^2+3-s^2}{s^2+1} = \frac{s^2+3}{s^2+1}$$

$$L[y(t)] = \frac{s^2+3}{(s^2+1)(1-s^2)}$$

$$\frac{s^2+3}{(s^2+1)(1-s^2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{1-s^2}$$

$$s^2+3 = (As+B)(1-s^2) + (Cs+D)(s^2+1)$$

equating s^3	equating s	equating s^2	put $s=0$
$0 = -A + C$	$0 = A + C$	$1 = -B + D$	$3 = B + D$

$$0 = -\cancel{A} + C$$
$$0 = \cancel{A} + C$$
$$0 = 2C$$
$$\boxed{C = 0 \quad A = 0}$$

$$1 = -B + D$$
$$3 = B + D$$
$$4 = 2D$$
$$\boxed{D = 2}$$

$$1 = -B + 2$$
$$B = 2 - 1$$
$$\boxed{B = 1}$$

$$\frac{s^2+3}{(s^2+1)(1-s^2)} = \frac{0+1}{s^2+1} + \frac{0+2}{1-s^2}$$

$$L[y(t)] = \frac{1}{s^2+1} + \frac{2}{1-s^2}$$

$$y(t) = L^{-1}\left[\frac{1}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$y(t) = \sin t - 2\sinh t$$

To find $x(t)$ we have

$$x(t) + y'(t) = \cos t$$

$$x(t) = \cos t - y'(t)$$

$$y(t) = \sin t - 2\sinh t$$

$$y'(t) = \cos t - 2\cosh t$$

$$x(t) = \cos t - [\cos t - 2\cosh t]$$

$$= \cancel{\cos t} - \cancel{\cos t} + 2\cosh t$$

$$x(t) = 2\cosh t$$

Hence $x(t) = 2\cosh t \qquad y(t) = \sin t - 2\sinh t$

③ Solve $Dx + x + Dy - y = 2$ and $D^2x + Dx - Dy = \cos t$ for which $x = 0$, $Dx = 2$ and $y = 1$ when $t = 0$

Soln:

Given

$$x'(t) + x(t) + y'(t) - y(t) = 2$$

$$x''(t) + x'(t) - y'(t) = \cos t$$

$$L[x'(t)] + L[x(t)] + L[y'(t)] - L[y(t)] = L(2)$$

$$L[x''(t)] + L(x'(t)) - L[y'(t)] = L[\cos t]$$

$$sL[x(t)] - x(0) + L[x(t)] + sL[y(t)] - y(0) - L[y(t)] = 2L(1)$$

$$(s+1)L[x(t)] + (s-1)L[y(t)] = 2/s + 1 \quad \text{——} ①$$

$$s^2L[x(t)] - sx(0) - x'(0) + sL[x(t)] - x(0) - sL[y(t)] + y(0) = \frac{s}{s^2+1}$$

$$(s^2+s)L[x(t)] - sL[y(t)] = \frac{s}{s^2+1} + 1 \quad \text{——} ②$$

Solving ① and ② we get

$$L[x(t)] = \frac{\begin{vmatrix} 2/s+1 & s-1 \\ \frac{s}{s^2+1}+1 & -s \end{vmatrix}}{\begin{vmatrix} s+1 & s-1 \\ s^2+s & -s \end{vmatrix}}$$

$$= \frac{(2/s+1)(-s) - (s-1)\left(\frac{s}{s^2+1}+1\right)}{(s+1)(-s) - (s-1)(s^2+s)}$$

$$= \frac{-2 - s - \left[\frac{s(s-1)}{s^2+1} + s - 1\right]}{-s^2 - s - [s^3 + s^2 - s^2 - s]}$$

$$= \frac{(-2-s)(s^2+1) - [(s^2-s) + (s-1)(s^2+1)] / (s^2+1)}{-s^2 - \cancel{s} - s^3 - \cancel{s^2} + \cancel{s^2} + \cancel{s}}$$

$$= \frac{-2s^2 - s^3 - 2 - s - [\cancel{s^2} - \cancel{s} + s^3 + \cancel{s} - \cancel{s^2} - 1] / (s^2+1)}{-s^2 - s^3}$$

$$= \frac{[-2s^2 - s^3 - 2 - s - s^3 + 1] / (s^2+1)}{-s^2(s+1)}$$

$$= \frac{-2s^2 - 2s^3 - s - 1}{-s^2(s+1)(s^2+1)} = \frac{-2s^2(1+s) - 1(s+1)}{-s^2(s+1)(s^2+1)} = \frac{\cancel{(s+1)}(-2s^2-1)}{-s^2\cancel{(s+1)}(s^2+1)}$$

$$= \frac{\cancel{-}(2s^2+1)}{\cancel{-}s^2(s^2+1)} = \frac{2s^2+1}{s^2(s^2+1)}$$

$$L[x(t)] = \frac{2s^2+1}{s^2(s^2+1)}$$

$$\frac{s^2+1+s^2}{s^2(s^2+1)} = \frac{2s^2+1}{s^2(s^2+1)}$$

$$x(t) = L^{-1}\left[\frac{2s^2+1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^2} + \frac{1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$x(t) = t + \sin t$$

$$L[y(t)] = \frac{\begin{vmatrix} s+1 & \frac{2}{s}+1 \\ s^2+s & \frac{s}{s^2+1}+1 \end{vmatrix}}{\begin{vmatrix} s+1 & s-1 \\ s^2+s & -s \end{vmatrix}}$$

$$= \frac{(s+1)\left[\frac{s+s^2+1}{s^2+1}\right] - (s^2+s)\overset{\left(\frac{2}{s}+1\right)}{\cancel{\left(\frac{2+s}{s}\right)}}}{(s+1)(-s) - (s-1)(s^2+s)}$$

$$= \frac{(s+1)\left[\frac{s^2+s+1}{s^2+1}\right] - \left[\cancel{\frac{2s^2+s^3+2s+s^2}{s}}\right] \left[\frac{2s^2}{s} + \frac{2s}{s} + s^2 + s\right]}{-s^2 - s - \left[s^3 + \cancel{s^2} - \cancel{s^2} - s\right]}$$

$$= \frac{(s+1)\left[\frac{s^2+s+1}{s^2+1}\right] - \left[2s+2+s^2+s\right]}{-s^2 \cancel{-s} - s^3 \cancel{+s}}$$

$$= \frac{(s+1)\left[\frac{s^2+s+1}{s^2+1}\right] - \left[s^2+3s+2\right]}{-s^2(s+1)}$$

$$= \frac{(s+1)\left[\frac{s^2+s+1}{s^2+1}\right] - \left[(s+1)(s+2)\right]}{-s^2(s+1)}$$

$$= \frac{\cancel{(s+1)}\left[s^2+s+1 - (s+2)(s^2+1)\right] / s^2+1}{-s^2\cancel{(s+1)}}$$

$$= \frac{s^2+s+1 - \left[s^3+2s^2+s+2\right]}{-s^2(s^2+1)} = \frac{s^2+\cancel{s}+1 - s^3 - 2s^2 \cancel{-s} - 2}{-s^2(s^2+1)}$$

$$\frac{-s^3-s^2-1}{-s^2(s^2+1)}$$

$$\frac{\cancel{-}(s^3+s^2+1)}{\cancel{-}s^2(s^2+1)}$$

$$L[y(t)] = \frac{s^3+s^2+1}{s^2(s^2+1)}$$

$$\frac{1}{s^2} + \frac{s}{s^2+1} = \frac{(s^2+1)+s(s^2)}{s^2(s^2+1)}$$

$$= \frac{s^2+1+s^3}{s^2(s^2+1)}$$

$$L[y(t)] = \frac{1}{s^2} + \frac{s}{s^2+1}$$

$$y(t) = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$y(t) = t + \cos t$$

$$\therefore x(t) = t + \sin t, \quad y(t) = t + \cos t$$