MA6459 NUMERICAL METHODS L T P C 3 1 0 4

OBJECTIVES:

This course aims at providing the necessary basic concepts of a few numerical methods and give procedures for solving numerically different kinds of problems occurring in engineering and technology

UNIT I SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS 10+3

Solution of algebraic and transcendental equations - Fixed point iteration method – Newton Raphson method- Solution of linear system of equations - Gauss elimination method – Pivoting - Gauss Jordan method – Iterative methods of Gauss Jacobi and Gauss Seidel - Matrix Inversion by Gauss Jordan method - Eigen values of a matrix by Power method.

UNIT II INTERPOLATION AND APPROXIMATION 8+3

Interpolation with unequal intervals - Lagrange's interpolation – Newton's divided difference interpolation – Cubic Splines - Interpolation with equal intervals - Newton's forward and backward difference formulae.

UNIT III NUMERICAL DIFFERENTIATION AND INTEGRATION 9+3

Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson's 1/3 rule – Romberg's method - Two point and three point Gaussian quadrature formulae – Evaluation of double integrals by Trapezoidal and Simpson's 1/3 rules.

UNIT IV INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS 9+3

Single Step methods - Taylor's series method - Euler's method - Modified Euler's method – Fourth order Runge-Kutta method for solving first order equations - Multi step methods - Milne's and Adams-Bash forth predictor corrector methods for solving first order equations.

UNIT V BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS 9+3

Finite difference methods for solving two-point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace's and Poisson's equations on rectangular domain – One dimensional heat flow equation by explicit and implicit (Crank Nicholson) methods –One dimensional wave equation by explicit method.

TOTAL (L:45+T:15): 60 PERIODS

OUTCOMES:

The students will have a clear perception of the power of numerical techniques, ideas and would be able to demonstrate the applications of these techniques to problems drawn from industry, management and other engineering fields.

TEXT BOOKS:

1. Grewal. B.S., and Grewal. J.S.,"Numerical methods in Engineering and Science", Khanna Publishers, 9th Edition, New Delhi, 2007.
2. Gerald. C. F., and Wheatley. P. O., "Applied Numerical Analysis", Pearson Education, Asia,
6th Edition, New Delhi, 2006.

REFERENCES:

1. Chapra. S.C., and Canale.R.P., "Numerical Methods for Engineers, Tata McGraw Hill, 5th Edition, New Delhi, 2007.
2. Brian Bradie. "A friendly introduction to Numerical analysis", Pearson Education, Asia, New Delhi, 2007.
3. Sankara Rao. K., "Numerical methods for Scientists and Engineers", Prentice Hall of India Private, 3rd Edition, New Delhi, 2007.

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

Method of False position (or) Regula Falsi method (or) linear interpolation method.

If $f(x_1) f(a) < 0$, then x_2 lies between x_1 and a

$$x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

① Find the positive root of $x^3 - 2x - 5 = 0$ by the Regula Falsi method.

Soln:

Let $f(x) = x^3 - 2x - 5 = 0$

There is only one poistive root by Descarte's rule of signs

$f(0) = -5 = -ve$

$f(1) = 1-2-5 = -6 = -ve$

$f(2) = 8-4-5 = -1 = -ve$

$f(3) = 27-6-5 = 16 = +ve$

Therefore the positive root lies between 2 and 3. It is closer to 2 also.

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2 f(3) - 3 f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)}$$

$$= \frac{32 + 3}{17} = \frac{35}{17}$$

$= 2.0588$ [correct to 4 decimal places]

$f(x_1) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$

$= 8.7265 - 4.1176 - 5$

$= -0.3911$

∴ The root lies between 2.0588 and 3

$$x_2 = \frac{2.0588\, f(3) - 3\, f(2.0588)}{f(3) - f(2.0588)}$$

$$= \frac{2.0588\,(16) - 3(-0.3911)}{16 - (-0.3911)}$$

$$= \frac{32.9408 + 1.1733}{16.3911} = \frac{34.1141}{16.3911}$$

$$= 2.0813$$

$$f(x_2) = f(2.0813) = (2.0813)^3 - 2(2.0813) - 5$$

$$= 9.0158 - 4.1626 - 5$$

$$= -0.1468$$

$\therefore$ The root lies between 2.0813 and 3

$$x_3 = \frac{2.0813\, f(3) - 3\, f(2.0813)}{f(3) - f(2.0813)}$$

$$= \frac{2.0813(16) - 3(-0.1468)}{16 - (-0.1468)}$$

$$= \frac{33.3008 + 0.4404}{16.1468} = \frac{33.7412}{16.1468}$$

$$= 2.08965$$

$$= 2.0897 \text{ (four decimal places)}$$

$$f(x_3) = f(2.0897) = (2.0897)^3 - 2(2.0897) - 5$$

$$= -0.054 = -ve$$

$\therefore$ The root lies between 2.0897 and 3

$$x_4 = \frac{(2.0897)\, f(3) - 3\, f(2.0897)}{f(3) - f(2.0897)}$$

$$= \frac{(2.0897)(16) - 3(-0.054)}{16 - (-0.054)}$$

$$= \frac{33.4352 + 0.162}{16.054}$$

$$= \frac{33.5972}{16.054} = 2.0928$$

$$f(x_4) = f(2.0928) = (2.0928)^3 - 2(2.0928) - 5$$
$$= 9.1661 - 4.1856 - 5$$
$$= -0.0195 = -ve$$

∴ The root lies between 2.0928 and 3

$$x_5 = \frac{2.0928\, f(3) - 3 f(2.0928)}{f(3) - f(2.0928)}$$

$$= \frac{(2.0928)(16) - 3(-0.0195)}{16 - (-0.0195)}$$

$$= \frac{33.4848 + 0.0585}{16.0195}$$

$$= \frac{33.5433}{16.995}$$

$$= 2.0939$$

$$f(x_5) = f(2.0939) = (2.0939)^3 - 2(2.0939) - 5$$
$$= 9.1805 - 4.1878 - 5$$
$$= -0.0073$$
$$= -ve$$

∴ The root lies between 2.0939 and 3

$$x_6 = \frac{(2.0939) f(3) - 3 f(2.0939)}{f(3) - f(2.0939)}$$

$$= \frac{(2.0939)(16) - 3(-0.0073)}{16 - (-0.0073)}$$

$$= \frac{33.5024 + 0.0219}{16.0073} = \frac{33.5243}{16.0073} = 2.0943$$

$$f(x_6) = f(2.0943) = (2.0943)^3 - 2(2.0943) - 5$$

$$= 9.1868 - 4.1886 - 5$$

$$= -0.0028$$

$$= -ve$$

$\therefore$ The root lies between 2.0943 and 3

$$x_7 = \frac{(2.0943)\, f(3) - 3\, f(2.0943)}{f(3) - f(2.0943)}$$

$$= \frac{(2.0943)(16) - 3(-0.0028)}{16 - (-0.0028)}$$

$$= \frac{33.5088 + 0.0084}{16.0028}$$

$$= \frac{33.5172}{16.0028} = 2.0945$$

$$f(x_7) = f(2.0945) = (2.0945)^3 - 2(2.0945) - 5$$

$$= 9.1884 - 4.189 - 5$$

$$= -0.0006 = -ve$$

$\therefore$ The root lies between 2.0945 and 3

$$x_8 = \frac{2.0945\, f(3) - 3\, f(2.0945)}{f(3) - f(2.0945)}$$

$$= \frac{(2.0945)(16) - 3(-0.0006)}{16 - (-0.0006)}$$

$$= \frac{33.512 + 0.0018}{16.0006}$$

$$= \frac{33.5138}{16.0006}$$

$$= 2.0945$$

We observe that $x_7 = x_8 = 2.0945$ correct to 4 places of decimals.

Hence the required root correct to four places of decimals is 2.0945

The results of the complete working are tabulated below.

Iteration(r)	a	b	x_r	sign of $f(x_r)$
1	2	3	2.0588	-0.3911
2	2.0588	3	2.0813	-0.1468
3	2.0813	3	2.0897	-0.054
4	2.0897	3	2.0928	-0.0195
5	2.0928	3	2.0939	-0.0073
6	2.0939	3	2.0943	-0.0028
7	2.0943	3	2.0945	-0.0006
8	2.0945	3	2.0945	

Formula $x_r = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

We observe that $x_7 = x_8 = 2.0945$

Hence the required root is 2.0945

② Using method of false position find a root of the equation $x^3 - 3x - 5 = 0$

Soln:

Given $f(x) = x^3 - 3x - 5$

$f(0) = 0 - 0 - 5 = -5 = -ve$

$f(1) = 1 - 3 - 5 = 1 - 8 = -7 = -ve$

$f(2) = 8 - 6 - 5 = 8 - 11 = -3 = -ve$

$f(3) = 27 - 9 - 5 = 27 - 14 = 13 = +ve$

$\therefore$ one root lies between 2 and 3

let $a = 2, \quad b = 3$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2(13) - 3(-3)}{13 - (-3)}$$

$$= \frac{26+9}{16} = \frac{35}{16} = 2.1875$$

$$f(x_1) = f(2.1875) = (2.1875)^3 - 3(2.1875) - 5$$

$$= 10.4675 - 6.5625 - 5$$

$$= -1.095$$

$\therefore$ one root lies between 2.1875 and 3

$$x_2 = \frac{(2.1875) f(3) - 3 f(2.1875)}{f(3) - f(2.1875)}$$

$$= \frac{(2.1875)(13) - 3(-1.095)}{13 - (-1.095)}$$

$$= \frac{28.4375 + 3.285}{14.095}$$

$$= \frac{31.7225}{14.095} = 2.2506$$

$$f(x_2) = f(2.2506) = (2.2506)^3 - 3(2.2506) - 5$$

$$= 11.3997 - 6.7518 - 5$$

$$= -0.3521 = -ve$$

$\therefore$ The root lies between 2.2506 and 3

$$x_3 = \frac{2.2506 f(3) - 3 f(2.2506)}{f(3) - f(2.2506)}$$

$$= \frac{(2.2506)(13) - 3(-0.3521)}{13 - (-0.3521)}$$

$$= \frac{29.2578 + 1.0563}{13.3521} = \frac{30.3141}{13.3521} = 2.2704$$

$$f(x_3) = f(2.2704) = (2.2704)^3 - 3(2.2704) - 5$$
$$= 11.7033 - 6.8112 - 5$$
$$= -0.1079 = -ve$$

$\therefore$ The root lies between 2.2704 and 3

$$x_4 = \frac{2.2704\, f(3) - 3 f(2.2704)}{f(3) - f(2.2704)}$$

$$= \frac{(2.2704)(13) - 3(-0.1079)}{13 - (-0.1079)}$$

$$= \frac{29.5152 + 0.3237}{13.1079} = \frac{29.8389}{13.1079} = 2.2764$$

$$f(x_4) = f(2.2764) = (2.2764)^3 - 3(2.2764) - 5$$
$$= 11.7963 - 6.8292 - 5$$
$$= -0.0329 = -ve$$

$\therefore$ The root lies between 2.2764 and 3

$$x_5 = \frac{(2.2764)\, f(3) - 3 f(2.2764)}{f(3) - f(2.2764)}$$

$$= \frac{(2.2764)(13) - 3(-0.03929)}{13 - (-0.0329)}$$

$$= \frac{25.5932 + 0.0987}{13.0329} = \frac{29.6919}{13.0329} = 2.2782$$

$$f(x_5) = f(2.2782) = (2.2782)^3 - 3(2.2782) - 5$$
$$= 11.8243 - 6.8346 - 5$$
$$= -0.0103 = -ve$$

$\therefore$ The root lies between 2.2782 and 3

$$x_6 = \frac{(2.2782)\, f(3) - 3 f(2.2782)}{f(3) - f(2.2782)}$$

$$= \frac{(2.2782)(13) - 3(-0.0103)}{13 - (-0.0103)}$$

$$= \frac{29.6166 + 0.0309}{13.0103} = \frac{29.6475}{13.0103} = 2.2788$$

$$f(x_6) = f(2.2788) = (2.2788)^3 - 3(2.2788) - 5$$

$$= 11.8336 - 6.8304 - 5$$

$$= -0.0028 = -ve$$

$\therefore$ The root lies between 2.2788 and 3

$$x_7 = \frac{2.2788\, f(3) - 3 f(2.2788)}{f(3) - f(2.2788)}$$

$$= \frac{(2.2788)(13) - 3(-0.0028)}{13 - (-0.0028)}$$

$$= \frac{29.6244 + 0.0084}{13.0028}$$

$$= \frac{29.6328}{13.0028}$$

$$= 2.2790$$

$$f(x_7) = f(2.2790) = (2.2790)^3 - 3(2.2790) - 5$$

$$= 11.8367 - 6.837 - 5$$

$$= -0.0003 = -ve$$

$\therefore$ The root lies between 2.279 and 3

$$x_8 = \frac{2.279\, f(3) - 3 f(2.279)}{f(3) - f(2.279)}$$

$$= \frac{(2.279)(13) - 3(-0.0003)}{13 - (-0.0003)}$$

$$= \frac{29.627 + 0.0009}{13.0003}$$

$$= \frac{29.6279}{13.0003}$$

$$= 2.2790$$

we observe that $x_7 = x_8 = 2.2790$ correct to four places of decimals.

Hence the required root is 2.2790.

TABLE

$f(x) = x^3 - 3x - 5$ Formula $x_r = \frac{af(b) - bf(a)}{f(b) - f(a)}$

Iteration(r)	a	b	x_r	$f(x_r)$
1	2	3	2.1875	
2	2.1875	3	2.2506	-1.095
3	2.2506	3	2.2704	-0.3521
4	2.2704	3	2.2764	-0.1079
5	2.2764	3	2.2782	-0.0329
6	2.2782	3	2.2788	-0.0103
7	2.2788	3	2.2790	-0.0028
8	2.2790	3	2.2790	-0.0003

Hence the required root is 2.2790

③ Find an approximate root of $x \log_{10} x - 1.2 = 0$ by Regula Falsi method.

Soln:- Let $f(x) = x \log_{10} x - 1.2$

$f(1) = 0 - 1.2 = -1.2 = -ve$

$f(2) = 2(0.30103) - 1.2 = -0.5979 = -ve$

$f(3) = 3 \cdot (0.47712) = 0.2314 = +ve$

Hence a root lies between 2 and 3

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)} = \frac{0.4628 + 1.7937}{0.8293}$$

$$= \frac{2.2565}{0.8293} = 2.7210$$

$$f(x_1) = f(2.7210) = (2.7210)\log_{10} 2.7210 - 1.2$$

$$= -0.0171 = -ve$$

Therefore the root lies between 2.7210 and 3

$$x_2 = \frac{(2.7210)\,f(3) - 3\,f(2.7210)}{f(3) - f(2.7210)}$$

$$= \frac{0.68094}{0.2485} = 2.7402$$

$$f(x_2) = f(2.7402) = (2.7402)\log_{10} 2.7402 - 1.2$$

$$= -0.0004 = -ve$$

$\therefore$ The root lies between 2.7402 and 3

$$x_3 = \frac{2.7402\,f(3) - 3\,f(2.7402)}{f(3) - f(2.7402)}$$

$$= \frac{(2.7402)(0.2314) - 3(-0.0004)}{(0.2314) - (-0.0004)}$$

$$= \frac{(2.7407)(0.2314) - 3(-0.0004)}{(0.2314) - (-0.0004)}$$

$$= 2.7407$$

$$f(x_3) = f(2.7407) = 2.7407\log_{10} 2.7407 - 1.2$$

$$= 0.0001 = +ve$$

$\therefore$ The root lies between 2.7402 and 2.7407

$$x_4 = \frac{2.7402\,f(2.7407) - (2.7407)\,f(2.7402)}{f(2.7407) - f(2.7402)}$$

$$= \frac{(2.7402)(0.0001) - (2.7407)(-0.0004)}{(0.0001) - (-0.0004)}$$

$$= \frac{(2.7402)(0.0001) + (2.7407)(0.0004)}{0.0001 + 0.0004}$$

$$= 2.7406$$

$$f(x_4) = f(2.7406) = 2.7406 \log_{10} 2.7406 - 1.2$$

$$= -0.0004 = -ve$$

$$x_5 = \frac{2.7406\, f(2.7407) - 2.7407\, f(2.7406)}{f(2.7407) - f(2.7406)}$$

$$= \frac{(2.7406)(0.0001) - (2.7407)(-0.00004)}{0.0001 - (-0.00004)}$$

$$= \frac{(2.7406)(0.0001) + (2.7407)(0.00004)}{0.0001 + 0.00004} = 2.7406$$

we observe that $x_4 = x_5 = 2.7406$

correct to four places of decimals

Hence the required root is 2.7406

$f(x) = x \log_{10} x - 1.2$		Formula $x_r = \frac{a f(b) - b f(a)}{f(b) - f(a)}$		
Iteration(r)	a	b	x_r	$f(x_r)$
1	2	3	2.7210	-0.0171
2	2.7210	3	2.7402	-0.0004
3	2.7402	3	2.7407	0.0001
4	2.7402	2.7407	2.7406	-0.00004
5	2.7406	2.7407	2.7406	-

we find that $f(2.7406)$ is approaching zero

Hence the required root is 2.7406.

NEWTON'S METHOD [Newton - Raphson method]

Formula:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

① Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton - Raphson method.

Soln:-

Let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = 1 - 1 - 10 = -10 = -ve$

$f(2) = 2^4 - 2 - 10 = 16 - 2 - 10 = 4 \ +ve$

∴ a root lies between 1 and 2.

Take $x_0 = 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left[\frac{2^4 - 2 - 10}{4(2)^3 - 1}\right]$$

$$= 2 - \left[\frac{4}{31}\right]$$

$$= 1.8709$$

$= 1.871$ [correct to three decimal places]

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{f(1.856)}{f'(1.856)}$$

$$= 1.856 - \left[\frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1}\right]$$

$$= 1.856 - \frac{0.010}{24.574}$$

$$= 1.856$$

The better approximate root is 1.856

② using Newton's iterative method find the root between 0 and 1 of $x^3 = 6x - 4$ correct to two decimal places.

Soln Let $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

$f(0) = 4 = +ve$

$f(1) = 1 - 6 + 4 = -1 = -ve$

$\therefore$ a root lies between 0 and 1

$|f(0)| > |f(1)|$

$\therefore$ This root is nearer to 1

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)} = 1 - \left[\frac{(1)^3 - 6(1) + 4}{3(1)^2 - 6}\right]$$

$$= 1 \frac{-(-1)}{-3} = 1 - 1/3 = 0.666$$

$= 0.67$ [Correct to two decimal places]

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.67 - \frac{f(0.67)}{f'(0.67)}$$

$$= 0.67 - \left[\frac{(0.67)^3 - 6(0.67) + 4}{3(0.67)^2 - 6}\right]$$

$$= 0.67 - \frac{0.28}{-4.65}$$

$$= 0.67 + \frac{0.28}{4.6\cancel{5}}$$

$$= 0.73$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73 - \frac{f(0.73)}{f'(0.73)}$$

$$= 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3(0.73)^2 - 6}\right]$$

$$= 0.73 - \left[\frac{0.009}{-4.4013}\right]$$

$$= 0.73 + \frac{0.009}{4.4013}$$

$$= 0.7320$$

$= 0.73$ [correct to two decimal places]

Here $x_2 = x_3 = 0.73$

$\therefore$ The root is 0.73 correct to two decimal places

(3) Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.

Soln

let $f(x) = 3x - \cos x - 1$

$f(0) = 0 - 1 - 1 = -2 = -ve$

$f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = 1.459698 = +ve$

$\therefore$ a root lies between 0 and 1

$|f(0)| > |f(1)|$

Hence the root is nearer to 1.

Let $x_0 = 0.6$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.6 - \frac{f(0.6)}{f'(0.6)}$$

$$= 0.6 - \left[\frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)}\right]$$

$$= 0.6 - (-0.007101)$$

$$= 0.607108$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.607108 - \left[\frac{3(0.607108) - \cos(0.607108) - 1}{3 + \sin(0.607108)}\right]$$

$$= 0.607108 - (0.000006)$$

$$= 0.607102$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.607102 - \frac{f(0.607102)}{f'(0.607102)}$$

$$= 0.607102 - \left[\frac{3(0.607102) - \cos(0.607102) - 1}{3 + \sin(0.607102)}\right]$$

$$= 0.607102 - 0.0000004$$

$$= 0.607102$$

Here $x_2 = x_3 = 0.607102$

$\therefore$ The root is 0.607102 correct to six decimals

(4) Find by NR method, the root of $x \log_{10} x = 12.34$ Start with $x_0 = 10$.

Soln:

Let $f(x) = x \log_{10} x - 12.34$

$$f'(x) = x \, \frac{1}{x} \log_{10} e + \log_{10} x$$

$$= \log_{10} e + \log_{10} x$$

Given $x_0 = 10$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 10 - \frac{f(10)}{f'(10)}$$

$$= 10 - \left[\frac{10 \log_{10} 10 - 12.34}{\log_{10} e + \log_{10} 10} \right]$$

$$= 10 - \left[\frac{-2.34}{1.4343} \right]$$

$$= 10 + \frac{2.34}{1.4343}$$

$$= 11.6315$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 11.6315 - \frac{f(11.6315)}{f'(11.6315)}$$

$$= 11.6315 - \left[\frac{11.6315 \log_{10} 11.6315 - 12.34}{\log_{10} e + \log_{10} 11.6315} \right]$$

$$= 11.6315 - \frac{0.0549}{1.5}$$

$$= 11.5949$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 11.5949 - \frac{f(11.5949)}{f'(11.5949)}$$

$$= 11.5949 - \left[\frac{11.5949 \log_{10} 11.5949 - 12.34}{\log_{10} e + \log_{10} 11.5949}\right]$$

$$= 11.5949 - \frac{0.00006}{1.4986}$$

$$= 11.5949$$

From x_2 and x_3 we find out the root is 11.5949.

FIXED POINT ITERATION $x = g(x)$ method.

(1) Solve the equation $x^2 - 2x - 3 = 0$ for the positive root by iteration method

Soln:
let $f(x) = x^2 - 2x - 3 = 0$

$f(x)$ is easy to factor to show roots at $x = -1$ and $x = 3$

Rearrange equation (1)

$$x = g(x) = \sqrt{2x+3}$$

let $x_0 = 4$

$$x_1 = g(x_0) = \sqrt{2x_0+3} = \sqrt{8+3} = \sqrt{11} = 3.31662$$

$$x_2 = g(x_1) = \sqrt{2x_1+3} = \sqrt{9.63325} = 3.10375$$

$$x_3 = g(x_2) = \sqrt{2x_2+3} = \sqrt{9.20750} = \sqrt{9.20750} = 3.03439$$

$$x_4 = g(x_3) = \sqrt{2x_3+3} = \sqrt{9.06877} = 3.01144$$

$$x_5 = g(x_4) = \sqrt{2x_4+3} = \sqrt{9.02288} = 3.00381$$

$$x_6 = g(x_6) = \sqrt{2x_6+3} = 3.00127$$

$$x_7 = g(x_7) = \sqrt{2x_7+3} = 3.00042$$

$$x_8 = g(x_7) = \sqrt{2x_7+3} = 3.00014$$

$$x_9 = g(x_8) = \sqrt{2x_8+3} = 3.00005$$

$$x_{10} = g(x_9) = \sqrt{2x_9+3} = 3.00002$$

$$x_{11} = g(x_{10}) = \sqrt{2x_{10}+3} = 3.00001$$

$$x_{12} = g(x_{11}) = \sqrt{2x_{11}+3} = 3.00000$$

$$x_{13} = g(x_{12}) = \sqrt{2x_{12}+3} = 3.00000$$

Here $x_{12} = x_{13} = 3$ [Correct 5 decimal places]

Hence the root is 3.

② Find a real root of the equation $x^3+x^2-100=0$

Soln:

Let $f(x) = x^3+x^2-100=0$

$f(0) = -100 = -ve$

$f(1) = 1+1-100 = -98 = -ve$

$f(2) = 8+4-100 = -88 = -ve$

$f(3) = 27+9-100 = -64 = -ve$

$f(4) = 64+16-100 = -20 = -ve$

$f(5) = 125+25-100 = 50 = +ve$

So there is a real root between 4 and 5

The given equation can be written as

$$x^2(x+1) = 100$$

$$x = \frac{10}{\sqrt{x+1}} = g(x)$$

$$g'(x) = \frac{10\,[-\frac{1}{2}]}{(x+1)^{3/2}} = \frac{-5}{(x+1)^{3/2}}$$

$$|g'(x)| = \frac{5}{(x+1)^{3/2}}$$

$$|g'(4)| = \frac{5}{5^{3/2}} < 1$$

$$|g'(5)| = \frac{5}{6^{3/2}} < 1$$

$\therefore$ $|g'(x)|$ is less than 1 in the interval $(4, 5)$

So the method can be applied

Let $x_0 = 4.2$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4.2+1}} = 4.38529$$

$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1+1}} = \frac{10}{\sqrt{4.38529+1}} = 4.30919$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{x_2+1}} = \frac{10}{\sqrt{4.30919+1}} = 4.33996$$

$$x_4 = g(x_3) = \frac{10}{\sqrt{x_3+1}} = \frac{10}{\sqrt{4.33996+1}} = 4.32744$$

$$x_5 = g(x_4) = \frac{10}{\sqrt{x_4+1}} = 4.33252$$

$$x_6 = g(x_5) = \frac{10}{\sqrt{x_5+1}} = 4.33046$$

$$x_7 = g(x_6) = \frac{10}{\sqrt{x_6+1}} = 4.33129$$

$$x_8 = g(x_7) = \frac{10}{\sqrt{x_7+1}} = 4.33096$$

$$x_9 = g(x_8) = \frac{10}{\sqrt{x_8+1}} = 4.33109$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{x_9+1}} = 4.33104$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{x_{10}+1}} = 4.33106$$

$$x_{12} = g(x_{11}) = \frac{10}{\sqrt{x_{11}+1}} = 4.33105$$

$$x_{13} = g(x_{12}) = \frac{10}{\sqrt{x_{12}+1}} = 4.33105$$

Here $x_{12} = x_{13} = 4.33105$ correct to 5 decimal places.

Gaussian Elimination method. & Gauss-Jordan method

① Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method.

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Soln:

(i) Gauss elimination method.

The given system is equivalent to

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

Here $[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$

Now, we will make the matrix A as a upper triangular

$\boxed{\text{Fix the first row, change 2 and 3 row with row 1}}$

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -32 & 91 & 341 \end{bmatrix} \quad \begin{matrix} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

$\boxed{\text{Fix 1 and 2 row, change 3 row with 2nd row}}$

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3 \leftrightarrow 52R_3 + 34R_2$$

This is an upper triangular matrix

From (1) we get [by back substitution]

$$3780\,z = 11340$$

$$\boxed{z = 3}$$

$$52y - 28z = -188$$

$$52y - 28(3) = -188$$

$$y = -2$$

$$10x - 2y + 3z = 23$$

$$10x - 2(-2) + 3(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x + 13 = 23$$

$$10x = 23 - 13$$

$$10x = 10$$

$$x = 1$$

Hence the solution is $x = 1,\ y = -2,\ z = 3$

(ii) <u>Gauss - Jordan method.</u>

Take the equation (1)

$$(A, B) \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad \begin{array}{l} R_1 \leftrightarrow 1260R_1 - R_3 \\ R_2 \leftrightarrow 135R_2 + R_3 \end{array}$$

Now we will make the matrix A a diagonal matrix

$\boxed{\text{Fix the third row and change 2nd row and first row}}$

$$\sim \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$\boxed{\text{Fix the 2 and 3 row change 1 row with 2 nd row}}$

$$\sim \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_1 \leftrightarrow 7020R_1 + 2520R_2$$

(ie) $$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \therefore x = 1,\ y = -2,\ z = 3$$

(2) solve the system of equations by Gauss - elimination method.

$$5x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5$$

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Soln:

The given system is equivalent to

$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -5 \\ -6 \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 4 & 29 & 4 & -29 \\ 0 & 4 & 4 & 19 & -34 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 5R_3 - R_1 \\ R_2 \leftrightarrow 5R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 0 & 970 & 120 & -1210 \\ 0 & 0 & 120 & 630 & -1380 \end{array}\right] \quad \begin{array}{l} R_3 \rightarrow 34R_3 - 4R_2 \\ R_4 \rightarrow 34R_4 - 4R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 12 & 63 & -138 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow \frac{R_2}{2} \\ R_3 \rightarrow \frac{R_3}{10} \\ R_4 \rightarrow \frac{R_4}{10} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 0 & 5967 & -11934 \end{array}\right] \quad R_4 \rightarrow 97R_4 - 12R_3$$

$$5967\, x_4 = -11934$$

$$x_4 = -2$$

$97x_3 + 12x_4 = -121$

$97x_3 + 12(-2) = -121$

$97x_3 - 24 = -121$

$97x_3 = -121 + 24$

$97x_3 = -97$

$\boxed{x_3 = -1}$

$17x_2 + 2x_3 + 2x_4 = 28$

$17x_2 + 2(-1) + 2(-2) = 28$

$17x_2 - 2 - 4 = 28$

$17x_2 - 6 = 28$

$17x_2 = 28 + 6$

$17x_2 = 34$

$\boxed{x_2 = 2}$

$5x_1 + x_2 + x_3 + x_4 = 4$

$5x_1 + 2 + (-1) + (-2) = 4$

$5x_1 - 1 = 4$

$5x_1 = 5$

$\boxed{x_1 = 1}$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = -2$

(3) using the Gauss-Jordan method solve the following equations

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Soln:

Interchanging the first and the last equation then

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right]$$

Fix the pivot element row and make the other elements zero in the pivot element column.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 10R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 + 9R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - 6.125R_3 \\ R_1 \leftrightarrow R_2 + 1.125R_3 \end{array}$$

$\therefore x_1 = 1,\ y = 1,\ z = 1$

ITERATIVE METHODS

(a) Gauss Jacobi method

(b) Gauss-Seidal method.

① Solve the following system of equations by Gauss-Jacobi method and Gauss-Seidal method.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Soln: As the coefficient matrix is not diagonally domiant we rewrite the equations.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

diagonally dominant

$\left[\because |a_{11}| > |a_{12}| + |a_{13}| ;\ |c_{33}| > |c_{31}| + |c_{32}| \right.$
$\left. |b_{22}| > |b_{21}| + |b_{23}| \right]$

Since the diagonal elements are dominant in the coefficient matrix we write x, y, z as follows:

$$x = \frac{1}{27}\left[85 - 6y + z\right]$$
$$y = \frac{1}{15}\left[72 - 6x - 2z\right]$$
$$z = \frac{1}{54}\left[110 - x - y\right]$$

(i) Gauss Jacobi method

Let the initial values be $x=0,\ y=0,\ z=0$

First iteration:-

$$x^{(1)} = \frac{1}{27}[85] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72] = 4.8$$

$$z^{(1)} = \frac{1}{54}[110] = 2.037$$

Second iteration:-

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6.(4.8) + (2.037)] = 2.157$$

$$y^{(2)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269$$

$$z^{(2)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 4.8] = 1.890$$

Third iteration:-

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.269) + 1.890] = 2.492$$

$$y^{(3)} = \frac{1}{15}[72 - 6x^{(2)} + 2z^{(2)}] = \frac{1}{15}[72 - 6(2.157) - 2(1.890)] = 3.685$$

$$z^{(3)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.157 - 3.269] = 1.937$$

Fourth iteration:-

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = \frac{1}{27}[85 - 6(3.685) + 1.937] = 2.401$$

$$y^{(4)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(3)}] = \frac{1}{15}[72 - 6(2.492) - 2(1.937)] = 3.545$$

$$z^{(4)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = \frac{1}{54}[110 - 2.492 - 3.685] = 1.923$$

Fifth iteration:-

$$x^{(5)} = \frac{1}{27}[85 - 6y^{(4)} + z^{(4)}] = \frac{1}{27}[85 - 6(3.545) + 1.923] = 2.432$$

$$y^{(5)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(4)}] = \frac{1}{15}[72 - 6(2.401) - 2(1.923)] = 3.583$$

$$z^{(5)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = \frac{1}{54}[110 - 2.401 - 3.645] = 1.927$$

Sixth Iteration

$$x^{(6)} = \frac{1}{27}[85 - 6y^{(5)} + z^{(5)}] = \frac{1}{27}[85 - 6(3.583) + 1.927] = 2.423$$

$$y^{(6)} = \frac{1}{15}[72 - 6x^{(5)} - 2z^{(5)}] = \frac{1}{15}[72 - 6(2.4332) - 2(1.927)] = 3.570$$

$$z^{(6)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right] = \frac{1}{54}\left[110 - 2.432 - 3.583\right] = 1.926$$

Seventh iteration:

$$x^{(7)} = \frac{1}{27}\left[85 - 6y^{(6)} + z^{(6)}\right] = \frac{1}{27}\left[85 - 6(3.570) + 1.926\right] = 2.426$$

$$y^{(7)} = \frac{1}{15}\left[72 - 6x^{(6)} - 2z^{(6)}\right] = \frac{1}{15}\left[72 - 6(2.423) - 2(1.926)\right] = 3.574$$

$$z^{(7)} = \frac{1}{54}\left[110 - x^{(6)} - y^{(6)}\right] = \frac{1}{54}\left[110 - 2.423 - 3.570\right] = 1.926$$

Eighth Iteration:

$$x^{(8)} = \frac{1}{27}\left[85 - 6y^{(7)} + z^{(7)}\right] = \frac{1}{27}\left[85 - 6(3.574) + 1.926\right] = 2.425$$

$$y^{(8)} = \frac{1}{15}\left[72 - 6x^{(7)} - 2z^{(7)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(8)} = \frac{1}{54}\left[110 - x^{(7)} - y^{(7)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.574\right] = 1.926$$

Nineth Iteration:

$$x^{(9)} = \frac{1}{27}\left[85 - 6y^{(8)} + z^{(8)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(9)} = \frac{1}{15}\left[72 - 6x^{(8)} - 2z^{(8)}\right] = \frac{1}{15}\left[72 - 6(2.425) - 2(1.926)\right] = 3.573$$

$$z^{(9)} = \frac{1}{54}\left[110 - x^{(8)} - y^{(8)}\right] = \frac{1}{54}\left[110 - 2.425 - 3.573\right] = 1.926$$

Tenth Iteration:

$$x^{(10)} = \frac{1}{27}\left[85 - 6y^{(9)} + z^{(9)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(10)} = \frac{1}{15}\left[72 - 6x^{(9)} - 2z^{(9)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(10)} = \frac{1}{54}\left[110 - x^{(9)} - y^{(9)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

[Correct to three decimal places]

(2) Gauss - seidel method.

Let the initial values be $y = 0$ $z = 0$.

First iteration

$$x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right] = \frac{1}{27}\left[85 - 6(0) + 0\right] = 3.148$$

$$y^{(1)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(0)}\right] = \frac{1}{15}\left[72 - 6(3.148) - 0\right] = 3.541$$

$$z^{(1)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right] = \frac{1}{54}\left[110 - 3.148 - 3.541\right] = 1.913$$

Second iteration:-

$$x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right] = \frac{1}{27}\left[85 - 6(3.541) + 1.913\right] = 2.432$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6x^{(2)} - 2z^{(1)}\right] = \frac{1}{15}\left[72 - 6(2.432) - 2(1.913)\right] = 3.572$$

$$z^{(2)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right] = \frac{1}{54}\left[110 - 2.432 - 3.572\right] = 1.926$$

Third iteration:

$$x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} + z^{(2)}\right] = \frac{1}{27}\left[85 - 6(3.572) + 1.926\right] = 2.426$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926$$

Fourth iteration:-

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

This shows that the convergence is rapid in Gauss-Seidel method when compared to Gauss-Jacobi method.

③ Solve the following equations by Gauss-Seidel method.

$$4x + 2y + z = 14$$

$$x + 5y - z = 10$$

$$x + y + 8z = 20$$

Soln:

as the coefficient matrix is diagonally dominant solving for x, y, z we get

$$x = \frac{1}{4}[14-2y-z]$$

$$y = \frac{1}{5}[10-x+z]$$

$$z = \frac{1}{8}[20-x-y]$$

let the initial values be $y=0$, $z=0$

First iteration:

$$x^{(1)} = \frac{1}{4}[14-2(0)-(0)] = \frac{14}{4} = 3.5$$

$$y^{(1)} = \frac{1}{5}[10-x^{(1)}+z^{(0)}] = \frac{1}{5}[10-3.5+0] = 1.3$$

$$z^{(1)} = \frac{1}{8}[20-x^{(1)}-y^{(1)}] = \frac{1}{8}[20-3.5-1.3] = 1.9$$

Second iteration:

$$x^{(2)} = \frac{1}{4}[14-2y^{(1)}-z^{(1)}] = \frac{1}{4}[14-2(1.3)-(1.9)] = 2.375$$

$$y^{(2)} = \frac{1}{5}[10-x^{(2)}+z^{(1)}] = \frac{1}{5}[10-2.375+1.9] = 1.905$$

$$z^{(2)} = \frac{1}{8}[20-x^{(2)}-y^{(2)}] = \frac{1}{8}[20-2.375-1.905] = 1.965$$

Third iteration:

$$x^{(3)} = \frac{1}{4}[14-2y^{(2)}-z^{(2)}] = \frac{1}{4}[14-2(1.905)-1.965] = 2.056$$

$$y^{(3)} = \frac{1}{5}[10-x^{(3)}+z^{(2)}] = \frac{1}{5}[10-2.056+1.965] = 1.9818$$

$$z^{(3)} = \frac{1}{8}[20-x^{(3)}-y^{(3)}] = \frac{1}{8}[20-2.056+1.9818] = 1.995$$

Fourth iteration:

$$x^{(4)} = \frac{1}{4}[14-2y^{(3)}-z^{(3)}] = \frac{1}{4}[14-2(1.9818)-1.965] = 2.510$$

$$y^{(4)} = \frac{1}{5}[10-x^{(4)}+z^{(3)}] = \frac{1}{5}[10-2.510+1.995] = 1.897$$

$$z^{(4)} = \frac{1}{8}[20-x^{(4)}-y^{(4)}] = \frac{1}{8}[20-2.510-1.897] = 1.949$$

Fifth iteration:-

$$x^{(5)} = \frac{1}{4}[14 - 2y^{(4)} - z^{(4)}] = \frac{1}{4}[14 - 2(1.897) - 1.949] = 2.064$$

$$y^{(5)} = \frac{1}{5}[10 - x^{(5)} + z^{(4)}] = \frac{1}{5}[10 - 2.064 + 1.949] = 1.977$$

$$z^{(5)} = \frac{1}{8}[20 - x^{(5)} - y^{(5)}] = \frac{1}{8}[20 - 2.064 - 1.977] = 1.995$$

Sixth iteration:-

$$x^{(6)} = \frac{1}{4}[14 - 2y^{(5)} - z^{(5)}] = \frac{1}{4}[14 - 2(1.977) - 1.995] = 2.013$$

$$y^{(6)} = \frac{1}{5}[10 - x^{(6)} + z^{(5)}] = \frac{1}{5}[10 - 2.013 + 1.995] = 1.996$$

$$z^{(6)} = \frac{1}{8}[20 - x^{(6)} - y^{(6)}] = \frac{1}{8}[20 - 2.013 - 1.996] = 1.999$$

Seventh iteration:-

$$x^{(7)} = \frac{1}{4}[14 - 2y^{(6)} - z^{(6)}] = \frac{1}{4}[14 - 2(1.996) - 1.999] = 2.002$$

$$y^{(7)} = \frac{1}{5}[10 - x^{(7)} + z^{(6)}] = \frac{1}{5}[10 - 2.002 + 1.999] = 1.999$$

$$z^{(7)} = \frac{1}{8}[20 - x^{(7)} - y^{(7)}] = \frac{1}{8}[20 - 2.002 - 1.999] = 2.000$$

Eighth iteration:-

$$x^{(8)} = \frac{1}{4}[14 - 2y^{(7)} - z^{(7)}] = \frac{1}{4}[14 - 2(1.999) - 2] = 2.001$$

$$y^{(8)} = \frac{1}{5}[10 - x^{(8)} + z^{(7)}] = \frac{1}{5}[10 - 2.001 + 2] = 2.000$$

$$z^{(8)} = \frac{1}{8}[20 - x^{(8)} - y^{(8)}] = \frac{1}{8}[20 - 2.001 - 2] = 2.000$$

Ninth iteration:-

$$x^{(9)} = \frac{1}{4}[14 - 2y^{(8)} - z^{(8)}] = \frac{1}{4}[14 - 2(2) - 2] = 2$$

$$y^{(9)} = \frac{1}{5}[10 - x^{(9)} + z^{(8)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(9)} = \frac{1}{8}[20 - x^{(9)} - y^{(9)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Tenth iteration:-

$$x^{(10)} = \frac{1}{4}[14 - 2y^{(9)} - z^{(9)}] = \frac{1}{4}[14 - 2(2) - 2] = 2$$

$$y^{(10)} = \frac{1}{5}[10 - x^{(10)} + z^{(9)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(10)} = \frac{1}{8}[20 - x^{(10)} - y^{(10)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Hence $x = 2, \; y = 2, \; z = 2$

INVERSE OF A MATRIX BY GAUSS JORDON METHOD

Gauss - Jordan elimination method.

① using Gauss-Jordan method, find the inverse of the matrix.

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

Soln:-

$$[A, I] = \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 1 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{-1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] \quad R_3 \leftrightarrow R_3(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 + \frac{1}{2}R_3 \\ R_2 \leftrightarrow R_2 - 2R_3 \end{array}$$

Hence $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$

verification:-

$$AA^{-1} = I$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) using Gauss - Jordan method, find the inverse of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Soln:

Let $[A, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$ $R_2 \leftrightarrow R_2 - R_1$, $R_3 \leftrightarrow R_3 + 2R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$ $R_2 \leftrightarrow \frac{R_2}{2}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_1 - R_2$, $R_3 \leftrightarrow R_3 + 2R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$ $R_3 \leftrightarrow \frac{R_3}{-4}$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$ $R_1 \leftrightarrow R_1 - 6R_3$, $R_2 \leftrightarrow R_2 + 3R_3$

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

Verification:-

$$AA^{-1} = I$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{Ans.}$$

EIGEN VALUE OF A MATRIX BY POWER METHOD

The power method

① Find the numerically largest eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

Soln:-

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial eigenvector.

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = 6X_2$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix} = 8.003 \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix} = 8.003\, X_3$$

$$0.167 - 3(0.067) + 2(1) = 0.166$$

$$4(0.167) + 4(0.667) - 1 = 2.336$$

$$6(0.167) + 3(0.667) + 5 = 8.003$$

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.145 \\ 0.252 \\ 6.002 \end{bmatrix} = 6.002 \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix} = 6.002\, X_4$$

$$1(0.021) - 3(0.292) + 2(1) = 1.145$$

$$4(0.021) + 4(0.292) - 1(1) = 0.252$$

$$6(0.021) + 3(0.292) + 5(1) = 6.002$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.065 \\ -0.068 \\ 6.272 \end{bmatrix} = 6.272 \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = 6.272\, X_5$$

$$1(0.191) - 3(0.042) + 2(1) = 2.065$$

$$4(0.191) + 4(0.042) - 1(1) = -0.068$$

$$6(0.191) + 3(0.042) + 5(1) = 6.272$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.362 \\ 0.272 \\ 6.941 \end{bmatrix} = 6.941 \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = 6.941\, X_6$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.223 \\ 0.516 \\ 7.157 \end{bmatrix} = 7.157 \begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix} = 7.157\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.095 \\ 0.532 \\ 7.082 \end{bmatrix} = 7.082 \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = 7.082\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.071 \\ 0.484 \\ 7.001 \end{bmatrix} = 7.001 \begin{bmatrix} 0.296 \\ 0.069 \\ 1 \end{bmatrix} = 7.001\, X_9$$

This shows the largest eigenvalue $= 7$.

② Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ find also the least latent root and hence the third value also.

Soln:

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximate eigen value.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7 \cdot X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714\, X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\, X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982\begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 3.9982\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4X_9$$

$\therefore$ Dominant eigen value $= 4$; corresponding eigen vector is $(1, 0.5, 0)$

To find the least eigen value, let $B = A - 4I$, since $\lambda_1 = 4$

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We will find the dominant eigen value of B.

let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5Y_3$$

$$\therefore BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value of B is -5

Adding 4, smallest eigen value of $A = -5 + 4 = -1$

Sum of eigen values $=$ trace of $A = 1 + 2 + 3 = 6$

$$4 + (-1) + \lambda_3 = 6, \quad \therefore \lambda_3 = 3$$

All the three eigen values are, 4, 3, −1.

UNIT-II

INTERPOLATION AND APPROXIMATION

LAGRANGIAN POLYNOMIALS

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

① Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	x_0 0	x_1 1	x_2 2	x_3 5
$f(x)$	2 y_0	3 y_1	12 y_2	147 y_3

Soln: By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x-2)(x-5)}{(-10)}(2) + \frac{x(x-2)(x-5)}{4}(3) + \frac{(x-1)(x-5)}{-6}(12) + \frac{x(x-1)(x-2)}{60}(147)$$

$$y = f(3) = \frac{(3-1)(3-2)(3-5)}{-10}(2) + \frac{3(3-2)(3-5)}{4}(3) + \frac{3(3-1)(3-5)}{-6}(12) + \frac{3(3-1)(3-2)}{60}(147)$$

$$y = \frac{(2)(1)(-2)}{(-10)}(2) + \frac{(3)(1)(-2)}{4}(3) + \frac{3(2)(-2)}{(-6)}(12) + \frac{(3)(2)(1)}{60}(147)$$

$$= \frac{4}{10}(2) - \frac{6}{4}(3) + 2(12) + \frac{1}{10}(147)$$

$$= \frac{8}{10} - \frac{18}{4} + 24 + \frac{147}{10}$$

$$= 35 \text{ Ans}$$

(2) Find the third degree polynomial $f(x)$ satisfying the following data.

x	1	3	5	7
y	24	120	336	720

Soln:
The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-3)(x-5)(x-7)}{(1-3)(1-5)(1-7)}(24) + \frac{(x-1)(x-5)(x-7)}{(3-1)(3-5)(3-7)}(120) + \frac{(x-1)(x-3)(x-7)}{(5-1)(5-3)(5-7)}(336)$$

$$+ \frac{(x-1)(x-3)(x-5)}{(7-1)(7-3)(7-5)}(720)$$

$$= -\tfrac{1}{2}(x-3)(x-5)(x-7) + \frac{15}{2}(x-1)(x-5)(x-7) - 21(x-1)(x-3)(x-7)$$

$$+ 15(x-1)(x-3)(x-5)$$

$$= -\tfrac{1}{2}[x^3-15x^2+71x-105] + \frac{15}{2}[x^3-13x^2+47x-35] - 21[x^3-11x^2+31x-21]$$

$$= +15[x^3-9x^2+23x-15]$$

$$= \left[-\tfrac{1}{2} + \frac{15}{2} - 21 + 15\right]x^3 + \left[\frac{15}{2} - \frac{195}{2} + 231 - 135\right]x^2 + \left[\frac{-71}{2} + \frac{705}{2} - 605 + 345\right]x$$

$$+ \left[\frac{105}{2} - \frac{525}{2} + 441 - 225\right]$$

$$= x^3 + 6x^2 + 11x + 6$$

$$f(4) = 4^3 + 6(4^2) + 11(4) + 6$$

$$= 64 + 96 + 44 + 6$$

$$= 210$$

③ using Lagrange's interpolation formula find $f(4)$ given that $f(0) = 2$, $f(1) = 3$, $f(2) = 12$, $f(15) = 3587$.

Soln:

Given

x	0	1	2	15
y	2	3	12	3587

By Lagrange's formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(4) = \frac{(4-1)(4-2)(4-15)}{(0-1)(0-2)(0-15)}(2) + \frac{(4-0)(4-2)(4-15)}{(1-0)(1-2)(1-15)}(3)$$

$$+ \frac{(4-0)(4-1)(4-15)}{(2-0)(2-1)(2-15)}(12) + \frac{(4-0)(4-1)(4-2)}{(15-0)(15-1)(15-2)} 3587$$

$$= \frac{(3)(2)(-11)}{(-1)(-2)(-15)}(2) + \frac{(4)(2)(-11)}{(1)(-1)(-14)}(3) + \frac{(4)(3)(-11)}{(2)(1)(-13)}(12)$$

$$+ \frac{(4)(3)(2)}{(15)(14)(13)}(3587)$$

$$= \frac{132}{30} - \frac{264}{14} + \frac{1584}{26} + \frac{86088}{2730}$$

$$= 78$$

④ Find the missing term in the following table using Lagrange's interpolation.

x	0	1	2	3	4
y	1	3	9	–	81

Soln:

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Let $x_0 = 0 \quad y_0 = 1$

$x_1 = 1 \quad y_1 = 3$

$x_2 = 2 \quad y_2 = 9$

$x_3 = 4 \quad y_3 = 81$

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3) + \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

$$f(3) = \frac{(3-1)(3-2)(3-4)}{(-1)(-2)(-4)}(1) + \frac{(3-0)(3-2)(3-4)}{(1)(-1)(-3)}(3) + \frac{(3-0)(3-1)(3-4)}{(2)(-2)(1)}(9) + \frac{(3-0)(3-1)(3-2)}{(4)(3)(2)}(81)$$

$$= \frac{(2)(1)(-1)}{(-1)(-2)(-4)}(1) + \frac{(3)(1)(-1)}{(1)(-1)(-3)}(3) + \frac{(3)(2)(-1)}{(2)(1)(-2)}(9) + \frac{(3)(2)(1)}{(4)(3)(2)}(81)$$

$$= \frac{-2}{-8}(1) - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= 31$$

⑤ Find the parabola of the form $y = ax^2 + bx + c$ passing through the points $(0,0)$ $(1,1)$ and $(2,20)$

Soln:

we use Lagrange's interpolation formula

$$y = f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 20.$$

$$= 0 - x(x-2) + 10x(x-1)$$

$$y = 9x^2 - 8x$$

Inverse interpolation

Taking y as independent variable

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

This is called formula of inversion interpolation.

① Find the age corresponding to the annuity value 13.6 given the table.

Age (x):	30	35	40	45	50
Annuity Value (y)	15.9	14.9	14.1	13.3	12.5

Soln:-

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)} \times 30 + \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \times 35 + \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} \times 40 + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \times 45 + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)} \times 50$$

$\therefore x(y=13.6) = 43$

② Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

Soln:
Now take $f(\theta)$ as independent and θ as dependent

$y = f(\theta)$:	0.3706	0.4068	0.4433
θ:	21	23	25

$$\theta = \frac{(y-0.4068)(y-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21 + \frac{(y-0.3706)(y-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23)$$

$$+ \frac{(y-0.3706)(y-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$\theta(y=0.3887) = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} \times 23$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$= 7.885832 + 17.202739 - 3.086525$$

$$= 22.0020$$

DIVIDED DIFFERENCES

TABLE

Argument x	Entry $f(x)$	First divided difference $\Delta f(x)$	Second divided difference $\Delta_2 f(x)$	Third divided difference $\Delta_3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$	
		$f(x_3, x_4)$		
x_4	$f(x_4)$			

① Form the divided difference table for the following data:-

x	1	2		4	7	12
$f(x)$	22	30		82	106	206

Soln:
The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-3.6-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{0.51+1.6}{12-1}=0.19$
		$\frac{106-82}{7-4}=8$		$\frac{1.5+3.6}{12-2}=0.51$	
7	106		$\frac{20-8}{12-4}=1.5$		
		$\frac{206-106}{12-7}=20$			
12	206				

(3) Show that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Soln:

If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$

$$f(a,b) = \underset{b}{\Delta}\left(\frac{1}{a}\right) = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \frac{1}{abc}\left(\frac{c-a}{c-a}\right) = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d)-f(a,b,c)}{d-a} = \frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = \frac{1}{abcd}\left(\frac{a-d}{d-a}\right) = -\frac{1}{abcd}$$

$$\therefore \underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$$

Newton's divided difference formula (or) Newton's interpolation for unequal intervals

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \dots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\dots,x_n)$$

① using Netwon's divided difference formula, find $u(3)$ given $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$.

Soln:

We form the divided difference table since the intervals are unequal.

x	$u(x)$	$\Delta u(x)$	$\Delta^2 u(x)$	$\Delta^3 u(x)$
1	-26			
		$\frac{12+26}{2-1} = 38$		
2	12		$\frac{122-38}{4-1} = 28$	
		$\frac{256-12}{4-2} = 122$		$\frac{43-28}{6-1} = 3$
4	256		$\frac{294-122}{6-2} = 43$	
		$\frac{844-256}{6-4} = 294$		
6	844			

By Newton's divided difference interpolation formula

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \cdots$

Here

$u(x) = u(x_0) + (x-x_0) u(x_0, x_1) + (x-x_0)(x-x_1) u(x_0, x_1, x_2) + \cdots$

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4, \quad x_3 = 6$

$u(x_0) = -26, \quad u(x_0, x_1) = 38 \quad u(x_0, x_1, x_2) = 28, \quad u(x_0, x_1, x_2, x_3) = 3$

$\therefore u(x) = -26 + (x-1)38 + (x-1)(x-2)28 + (x-1)(x-2)(x-4)3$

$\therefore u(3) = -26 + (2)(38) + (2)(1)(28) + (2)(1)(-1)(3)$

$= -26 + 76 + 56 - 6$

$u(3) = 100.$

② Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula.

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Soln:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{(-1)-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
		$\frac{5-33}{0-(-1)} = -28$		$\frac{10-94}{2-(-4)} = -14$	
0	5		$\frac{2-(-28)}{2-(-1)} = 10$		$\frac{13+14}{5-(-4)} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5-(-1)} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$

Here $x_0 = -4 \quad x_1 = -1 \quad x_2 = 0 \quad x_3 = 2 \quad x_4 = 5$

$f(x_0) = 1245$

$f(x_0, x_1) = -404$

$f(x_0, x_1, x_2) = 94$

$f(x_0, x_1, x_2, x_3) = -14$

$f(x_0, x_1, x_2, x_3, x_4) = 3$

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x)(-14) + (x+4)(x+1)(x)(x-2)(3)$$

$$= 1245 - 404x - 1616 + 94[x^2+5x+4] - 14x[x^2+5x+4] + 3x[(x^2+5x+4)(x-2)]$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x[x^3 - 2x + 5x^2 - 10x + 4x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x[x^3 + 5x^2 - 8x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x^4 + 15x^3 - 24x^2 - 24x = 3x^4 + x^3 - 14x + 5$$ Ans

(3) using Newton's divided difference formula find the missing value from the table.

x	1	2	4	5	6
y	14	15	5	-	9

Soln:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	14			
		$\frac{15-14}{2-1}=1$		
2	15		$\frac{-5-1}{4-1}=-2$	
		$\frac{5-15}{4-2}=-5$		$\frac{\frac{7}{4}+2}{6-1}=\frac{15/4}{5}=\frac{3}{4}$
4	5		$\frac{2+5}{6-2}=\frac{7}{4}$	
		$\frac{9-5}{6-4}=2$		
6	9			

$$f(x) = f(x_0) + (x-x_0)\,f(x_0,x_1) + (x-x_0)(x-x_1)\,f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\,f(x_0,x_1,x_2,x_3) + \cdots$$

$$= 14 + (x-1)(1) + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\left(\tfrac{3}{4}\right)$$

$$= 14 + x - 1 - 2(x-1)(x-2) + \tfrac{3}{4}(x-1)(x-2)(x-4)$$

$$f(5) = 13 + 5 - 2(4)(3) + \tfrac{3}{4}(4)(3)(1)$$

$$= 18 - 24 + 9 = 3$$

INTERPOLATION WITH A CUBIC SPLINE

Formula:

$$f_i(x) = \frac{f_i''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f_i''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3$$
$$+ \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x)$$
$$+ \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1}) \quad \text{---(1)}$$

This equation contains only two unknowns - the second derivatives at the end of each interval.

These unknowns can be evaluated using the following equation.

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right] + \frac{6}{x_i - x_{i-1}}\left[f(x_{i-1}) - f(x_i)\right] \quad \text{--- (2)}$$

(1) From the following table.

x	x_{i-1} 1	x_i 2	x_{i+1} 3
y	-8	-1	18

Comput $y(1.5)$ and $y'(1)$ using cubic spline

Soln:

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right] + \frac{6}{x_i - x_{i-1}}\left[f(x_{i-1}) - f(x_i)\right] \quad \text{--- (2)}$$

$$(2-1) f''(1) + 2(3-1) f''(2) + (3-2) f''(3) = \frac{6}{(3-2)}(18+1) + \frac{6}{2-1}[-8+1]$$

$$f''(1) + 4f''(2) + f''(3) = 6(19) + 6(-7)$$

$f''(1) = 0 \quad f''(3) = 0$ at the end points, $f'(3) = 0$, $4f''(2) = 72$

$$f''(2) = 18$$

From (1) we get

$$f(x) = 0 + \frac{1}{6}\frac{f''(2)}{(2-1)}(x-1)^3 + \left[\frac{-8}{2-1} - 0\right](2-x) + \left[\frac{-1}{2-1} - \frac{18}{6}(2-1)\right](x-1)$$

$$= \frac{1}{6} 18(x-1)^3 + (-8).(2-x) + [-1-3][x-1]$$

$$= 3(x-1)^3 - 8(2-x) - 4(x-1)$$

$$= 3(x-1)^3 - 16 + 8x - 4x + 4$$

$$= 3[x^3 - 3x^2 - 3x - 1] - 16 + 4x + 4 = 3x^3 - 9x^2 + 13 - 15$$

$$= 3x^3 - 9x^2 - 8$$

$$y(1.5) = f(1.5) = 3(0.5)^3 + 4(1.5) - 12 = -\frac{45}{8}$$

$$y' = f'(x) = 9(x-1)^2 + 4$$

$$y'(1) = f'(1) = 4 \text{ Ans.}$$

Another method

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \cdots$$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \quad \text{for } i = 1, 2, 3 \cdots (n-1)$$

① From the following table:

x	1	2	3
y	-8	-1	18

Compute $y(1.5)$ and $y'(1)$ using cubic spline

Soln:
Here $h = 1$, and $n = 2$. also assume $M_0 = 0$ and $M_2 = 0$

we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \quad \text{for } i = 1, 2 \cdots (n-1)$$

From this

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$\therefore 4M_1 = 6[-8 - 2(-1) + 18] = 72$$

$$\therefore M_1 = 18$$

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \cdots ⑥$$

For $1 \le x \le 2$ putting $i = 1$ we get

$$S(x) = \frac{1}{6}\left[18(x-1)^3\right] + (2-x)(-8) - 4(x-1)$$

$$= 3(x-1)^3 + 4x - 12 = 3x^3 - 9x^2 + 13x - 15$$

$y(1\cdot5) = S(1\cdot5) = 3(0\cdot5)^3 + 4(1\cdot5) - 12 = -\frac{45}{8}$

$y' = S'(x) = 9(x-1)^2 + 4$

$y'(1) = 4.$

② Given the points $(0, 0)$ $(\pi/2, 1)$ and $(\pi, 0)$ satisfying the function $y = \sin x$ $(0 \le x \le \pi)$ determine the value of $y(\pi/6)$ using the cubic spline approximation.

x	0	$\pi/2$	π
y	0	1	0

Soln:-

Here $h = \pi/2$ $n = 2$, also assume $M_0 = 0$. $M_2 = 0$

we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \text{ for } i = 1, 2, \dots (n-1)$$

From this

$$M_0 + 4M_1 + M_2 = \frac{6}{(\pi/2)^2}\left[0 - 2 + 0\right] = \frac{-48}{\pi^2}$$

$$4M_1 = \frac{-48}{\pi^2}$$

$$M_1 = \frac{-48}{\cancel{12}\,\pi^2} = \frac{-12}{\pi^2}$$

In the interval $[0, \pi/2]$, the national cubic spline is given by

$$S_1(x) = \frac{1}{6(\pi/2)}\left[(x-0)^3\left(\frac{-12}{\pi^2}\right) + \frac{1}{(\pi/2)}\left[\pi/2 - x\right]\left[0 - \frac{(\pi/2)^2}{6}0\right] + \frac{1}{(\pi/2)}\left[x-0\right]\right.$$
$$\left.\left[1 - \frac{(\pi/2)^2}{6}\left(\frac{-12}{\pi^2}\right)\right]\right]$$

$$= \frac{1}{3\pi}\left[x^3\left(\frac{-12}{\pi^2}\right) + \frac{2}{\pi}(x)\left[1 + 1/2\right]\right]$$

$$= \frac{1}{3\pi}\left[\frac{-12}{\pi^2}x^3\right] + \frac{3}{\pi}x$$

$$= \frac{1}{\pi}\left[\frac{-4}{\pi^2}x^3\right] + \frac{3}{\pi}x. = \frac{2}{\pi}\left[\frac{-2}{\pi^2}x^3 + \frac{3}{2}x\right]$$

$$y(\pi/6) = \frac{2}{\pi}\left[\frac{-\pi}{108} + \frac{\pi}{4}\right] = 0.4815$$

NEWTON FORWARD AND BACKWARD DIFFERENCE FORMULA

Forward interpolation formula.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x-x_0}{h}$

(1) using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

Soln: we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$(x_0)\,4$	$(y_0)\,1$			
		$3-1=2(\Delta y_0)$		
			$5-2=3(\Delta^2 y_0)$	
$(x_1)\,6$	$(y_1)\,3$			$-3-3=-6(\Delta^3 y_0)$
		$8-3=5(\Delta y_1)$		
			$2-5=-3(\Delta^2 y_1)$	
$(x_2)\,8$	$(y_2)\,8$			
		$10-8=2(\Delta y_2)$		
$(x_3)\,10$	$(y_3)\,10$			

There are only 4 data given. Hence the polynomial will be degree 3.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0.$$

where $u = \frac{x-x_0}{h}$ Here $x_0 = 4$, $h = 6-4 = 2$ [difference]

$$y(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{3!}(-6)$$

$$= 1 + x - 4 + \frac{(x-4)(x-6)}{8}(3) + \frac{(x-4)(x-6)(x-8)}{(8)(6)}(-6).$$

$$= x - 3 + \tfrac{3}{8}(x-4)(x-6) - \tfrac{1}{8}(x-4)(x-6)(x-8)$$

$$= x - 3 + \tfrac{3}{8}[x^2 - 10x + 24] - \tfrac{1}{8}[(x^2 - 10x + 24)(x-8)]$$

$= x-3+\frac{3}{8}\left[x^2-10x+24\right]-\frac{1}{8}\left[x^3-10x^2+24x-8x^2+80x-192\right]$

$= \frac{1}{8}\left[8x-24+x^2-10x+24-x^3+10x^2-24x+8x^2-80x+192\right]$

$= \frac{1}{8}\left[-x^3+19x^2-106x+192\right]$

$y(5) = \frac{1}{8}\left(-(5)^3+19(5)^2-106(5)+192\right]$

$= \frac{1}{8}\left[-125+475-530+192\right]$

$= \frac{1}{8}\left[12\right]$

$y(5) = 1.5.$

② A third degree polynomial passes through the points $(0,-1)$ $(1,1)$ $(2,1)$ and $(3,-2)$ using Newton's forward interpolation formula find the polynomial. Hence find the value at 1.5

Soln

we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
(x_0) 0	(y_0) -1	$1+1=2(\Delta y_0)$		
			$0-2=-2(\Delta^2 y_0)$	
(x_1) 1	(y_1) 1			$-3+2=-1(\Delta^3 y_0)$
		$1-1=0(\Delta y_1)$		
(x_2) 2	(y_2) 1		$-3-0=-3(\Delta^2 y_1)$	
		$-2-1=-3(\Delta y_2)$		
(x_3) 3	(y_3) -2			

There are only 4 data given. Hence the polynomial of degree 3.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x-x_0}{h}$

$x_0 = 0$, $h = 1-0 = 1$ (difference)

$\therefore u = x$

$$y(x) = -1 + \frac{x}{1!}(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$$= -1 + 2x - x(x-1) - \frac{1}{6} x(x-1)(x-2)$$

$$= -1 + 2x - x^2 + x - \frac{1}{6} x[x^2 - 3x + 2]$$

$$= -x^2 + 3x - 1 - \frac{1}{6}[x^3 - 3x^2 + 2x]$$

$$= \frac{1}{6}[-6x^2 + 18x - 6 - x^3 + 3x^2 - 2x]$$

$$= \frac{1}{6}[-x^3 - 3x^2 + 16x - 6]$$

$$= -\frac{1}{6}[x^3 + 3x^2 - 16x + 6]$$

$$y(1.5) = -\frac{1}{6}[(1.5)^3 + 3(1.5)^2 - 16(1.5) + 6]$$

$$= -\frac{1}{6}[3.375 + 6.75 - 24 + 6]$$

$$= -\frac{1}{6}[-7.875]$$

$$y(1.5) = 1.3125$$

③ From the data given below, find the number of students whose weight is between 60 to 70.

Weight in lbs:	0-40	40-60	60-80	80-100	100-120
No. of students:	250	120	100	70	50

Soln:

Difference table

x weight	y (No. of students)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470				20
			-30		
		70		10	
Below 100	540				
			-20		
		50			
Below 120	590				

Let us calculate the number of students whose weight is less than 70.

We will use forward difference formula.

$$u = \frac{x-x_0}{h} = \frac{70-40}{20} = 1.5$$

$$y(70) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \cdots$$

$$= 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(-10)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$= 423.59$$

$$\simeq 424.$$

Number of students whose weight is between 60 and 70.

$$= y(70) - y(60) = 424 - 370 = 54.$$

<u>Newton's backward interpolation formula</u>

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}$$

$$\nabla^4 y_n + \cdots$$

where $v = \frac{x-x_n}{h}$

① Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.07181250$, $f(-0.5) = -0.024750$

$f(-0.25) = 0.33493750$, $f(0) = 1.10100$. Hence find $f(-\frac{1}{3})$

Soln:-

Newton's backward difference formula is

$$y(x) = y_3 + \frac{v}{1!}\nabla y_3 + \frac{v(v+1)}{2!}\nabla^2 y_3 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_3$$

where $v = \frac{x-x_3}{h}$

Here we for the difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
(x_0) (-0.75)	y_0 -0.07181250			
		0.0470625		
(x_1) -0.5	y_1 -0.024750		0.312625	
				$(\nabla^3 y_3)$ 0.09375
		0.3596875		
(x_2) -0.25	y_2 0.33493750			
			$(\nabla^2 y_3)$ 0.400375	
(x_3) 0	y_3 1.10100	(∇y_3) 0.7660625		

Here $x_3 = 0$ $h = 0.25$ $v = \frac{x}{0.25} = \frac{x}{(1/4)} = 4x$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.406375)$$

$$+ \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.10100 + 3.06425 + 0.81275x(4x+1) + 0.0625x(4x+1)(4x+2)$$

$$= 1.101 + 3.06425x^2 + 0.81275x + 0.0625x[16x^2 + 12x + 2]$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x.$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101$$

To find $f(-1/3)$.

$$y(-1/3) = (-1/3)^3 + (4.001)(-1/3)^2 + 4.002(-1/3) + 1.101$$

$$= -\frac{1}{27} + 4.001(1/9) - 4.002(1/3) + 1.101$$

$$= 0.174518518 \text{ Ans}$$

② From the following table find the value of $\tan(0.28)$

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2533	0.3093

Soln: Let us form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

Since 0.28 lies in the end of the table, let us use Newton's backward interpolation formula.

$$f(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

where $v = \dfrac{x - x_n}{h} = \dfrac{0.28 - 0.30}{0.05} = -0.04 \quad [\because x_n = 0.30]$

$$y = 0.3093 + \frac{(-0.4)}{1!}(0.0540) + \frac{(-0.4)(-0.4+1)}{2!}(0.0014)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(0.0004)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}(0.0002)$$

$$y = 0.309 - 0.0216 - 0.000168 - 0.0000256 - 0.00000832$$

$$y = 0.28720.$$

Numerical Differentiation and Integration

Derivatives from Difference tables - divided differences and Finite Differences

Newton's forward difference formula

Newton's forward difference interpolation formula is

$$y(x_0 + uh) = y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $y(x)$ is a polynomial of degree n in x and $u = \frac{x - x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's backward difference formula

Newton's backward difference formula is

$$y(x) = y(x_n + vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \frac{x - x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Derivative using Stirling formula

The Stirling's formula is

$$y(x) = y_0 + \frac{u}{2}\left[\Delta y_0 + \Delta y_{n-1}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3 - u}{12}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \frac{u^4 - u^2}{24}\Delta^4 y_{-2} + \cdots$$

when $u = \frac{x - x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \cdots\right]$$

Derivative using Bessel's formula

$$y(x) = \frac{1}{2}[y_0 + y_1] + (u - \tfrac{1}{2})\Delta y_0 + \frac{u(u-1)}{4}(\Delta^2 y_{-1} + \Delta^2 y_0)$$
$$+ \frac{u(u-\frac{1}{2})(u-1)}{6}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{48}(\Delta^4 y_{-2} + \Delta^4 y_{-1})$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{(3u^2 - 3u + \frac{1}{2})}{6}\Delta^3 y_{-1} + \cdots\right]$$

Maxima and minimal of a tabulated function

For maxima or minima $\frac{dy}{dx} = 0$ Hence equating the right hand side of (1) to zero and retaining only upto third differences we obtain

$$\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 = 0$$

(ie) $\left(\frac{1}{2}\Delta^3 y_0\right)u^2 + (\Delta^2 y_0 - \Delta^2 y_0)u + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{2}\Delta^3 y_0\right) = 0$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table, we solve this quadratic form. Then the corresponding values of x are given by $x = x_0 + uh$ at which y is maximum (or) minimum.

Problems:-

① Find $f'(3)$ and $f''(3)$ for one following data:

x :	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	−14	−10.032	−5.296	−0.256	6.672	14

Solution:

Since we require $f'(3)$ and $f''(3)$ we use Newton's forward formula.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	−14					
		3.968				
3.2	−10.032		0.768			
		4.736		−0.464		
3.4	−5.296		0.304		2.048	
		5.04		1.584		−5.12
3.6	−0.256		1.888		−3.072	
		6.928		−1.488		
3.8	6.672		0.4			
		7.328				
4.0	14					

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Here $h = 0.2$

$$= \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.12)\right]$$

$$= \frac{1}{0.2}\left[3.968 - 0.384 - 0.1547 - 0.512 + 1.024\right]$$

$$= \frac{1}{0.2}\left[1.8933\right]$$

$$= 9.4665$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.768 - (-0.464) + \frac{11}{12}(2.048) - \frac{5}{6}(-5.12)\right]$$

$$= \frac{1}{0.04}\left[0.768 + 0.464 + 1.8773 + 4.267\right]$$

$$= \frac{1}{0.04}\left[7.3763\right]$$

$$= 184.4075$$

② Compute $f'(0)$ and $f''(4)$ from the data.

x	0	1	2	3	4
y	1	2.718	7.381	20.086	54.598

Solution:-

Since we require $f'(0.5)$ and $f''(3.5)$ we use Newton's forward formula and Newton's backward formula.

Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		1.718			
1	2.718		2.945		
		4.663		5.097	
2	7.381				8.668
			8.042		
		12.705			
3	20.0826			13.765	
			21.807		
		34.512			
4	54.598				

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[1.718 - \frac{1}{2}(2.945) + \frac{1}{3}(5.097) - \frac{1}{4}(8.668)\right]$$

$$= \left[1.718 - 1.4725 + 1.699 - 2.167\right]$$

$$= -0.2225$$

By Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[(34.512) + \frac{1}{2}(21.807) + \frac{1}{3}(13.765) + \frac{1}{4}(8.668)\right]$$

$$= 34.512 + 10.9035 + 4.588 + 2.167$$

$$= 52.1705$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[21.807 + 13.765 + \frac{11}{12}(8.668)\right]$$

$$= 21.807 + 13.765 + 7.9457$$

$$= 43.5177 \text{ m/s}$$

③ Find the maximum and minimum value of y tabulated below.

x	-2	-1	0	1	2	3	4
y	2	-0.25	0	-0.25	2	15.75	56

Solution:-

Newton's forward difference formula is

$$y(x) = \frac{1}{h}\left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \right]$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \dots \right]$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.5			
				-3		
		0.25			6	
			-0.5			
0	0					0
				3		
		$\boxed{-0.25}$				
					6	
1	-0.25		$\boxed{2.5}$			
						0
		2.25		$\boxed{9}$		
2	2				$\boxed{6}$	
			11.5			
		13.75				
				15		
3	15.75		26.5			
		40.25				
4	56					

Choosing $x_0 = 0$, $u = \frac{x-0}{1} = x$

$$\frac{dy}{dx} = \frac{1}{1}\left[-0.25 + \frac{(2x-1)}{2}(2.5) + \frac{3x^2-6x+2}{6}(9) + \frac{4x^3-18x^2+22x-6}{24}(6) \right]$$

$$= -0.25 + \frac{(2x-1)}{2}(2.5) + \frac{3x^2-6x+2}{6}(9) + \frac{4x^3-18x^2+22x-6}{24}(6)$$

$$= -0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5$$

$$\frac{dy}{dx} = x^3 - x$$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 - x = 0$

$$x(x^2-1) = 0$$

$$x = 0, \quad x^2 - 1 = 0$$

$$x = 0, \quad (x-1)(x+1) = 0$$

$$x = 0, \quad x = 1, \quad x = -1$$

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

at $x = 0$, $\frac{d^2y}{dx^2} = -1 = -ve$

$x = 1$ $\quad \frac{d^2y}{dx^2} = 3 - 1 = 2 = +ve$

$x = -1$ $\quad \frac{d^2y}{dx^2} = 3 - 1 = 2 = +ve$

$\therefore$ y is maximum at $x = 0$ minimum at $x = 1$ and -1

$$\therefore y(x) = \frac{1}{h}\left[y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots\right]$$

$$y(0) = \frac{1}{1}[0 + 0] = 0$$

$\therefore$ maximum value $= 0$

$$y(1) = \frac{1}{1}[y_0 + \Delta y_0 + 0 + 0 + \cdots]$$

$$= [0 + (-0.25)]$$

$$= -0.25$$

$\therefore$ maximum at $x = 1$, $y(1) = -0.25$.

④ Consider the following table of data:

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find $f'(0.25)$ using Newton's forward difference approximation $f'(0.6)$ using Stirling's approximation and $f'(0.95)$ using Newton's backward difference approximation.

Soln:-

Here $h = 0.2$

Newton's forward interpolation formula for derivatives

7

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \dots\right]$$

Where $u = \frac{x-x_0}{h}$ $\quad x = 0.25,\ x_0 = 0.2,\ h = 0.2 \quad u = \frac{0.25-0.2}{0.2} = 0.25$

The difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.9798652 (y_{-2})				
		-0.0620942 Δy_{-2}			
0.4	0.9177710 (y_{-1})		-0.047642 $\Delta^2 y_{-2}$		
		-0.1097362 Δy_{-1}		-0.0120473 $\Delta^3 y_{-2}$	
0.6	0.8080348 (y_0)		-0.0596893 $\Delta^2 y_{-1}$		-0.01310985 $\Delta^4 y_{-2}$
		-0.1694255 Δy_{0}		-0.02515715 $\Delta^3 y_{-1}$	
0.8	0.6386093 (y_1)		-0.08484645 $\Delta^2 y_0$		
		-0.25427195 Δy_1			
1.0	0.38433735 (y_2)				

$$y'(0.25) = \frac{1}{0.2}\Big[(-0.0620942) + \frac{(2(0.25)-1)}{2}(-0.047642)$$
$$+ \frac{3(0.25)^2 - 6(0.25)+2}{6}(-0.0120473)$$
$$+ \frac{4(0.25)^3 - 18(0.25)^2 + 22(0.25) - 6}{24}(-0.01310985)\Big]$$

$$= \frac{1}{0.2}\left[-0.0620942 + 0.0119105 - 0.001380419 + 0.000853505\right]$$

$$= \frac{1}{0.2}\left[-0.050710613\right]$$

$$= -0.253553065$$

$$= -0.2536$$ [correct to four decimal places]

Stirling's formula for derivative is

$u = \frac{x-x_0}{h}$, $\quad x = 0.6,\ x_0 = 0.2,\ h = 0.2$

$$u = \frac{0.6-0.2}{0.2} = \frac{0.4}{0.2} = 0.2$$

$$y'(x) = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\,\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{4u^3-2u}{4!}\Delta^4 y_{-2} + \cdots\right]$$

$$y'(0.6) = \frac{1}{0.2}\left[\frac{-0.1694255 - 0.1097362}{2} + (0.2)(-0.00596893) + \frac{3(0.2)^2-1}{6}\left[\frac{-0.02515715 - 0.0120473}{2}\right] + \frac{4(0.2)^3 - 2(0.2)}{24}(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.13958085 - 0.01193786 + (-0.14667)(-0.01860225) + (-0.01533)(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.13958085 - 0.01193786 + 0.0027\,28388 + 0.000200974\right]$$

$$= \frac{1}{0.2}\left[-0.148589348\right]$$

$$= -0.74294674$$

$= -0.74295$ correct to five decimal places

Newton's backward difference formula

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{2u^2+6u+2}{6}\nabla^3 y_n + \frac{2u^3+9u^2+11u+3}{12}\nabla^4 y_n + \cdots\right]$$

$$u = \frac{x - x_n}{h} = \frac{0.95-1}{0.2} = -0.25$$

$$y'(0.95) = \frac{1}{h}\left[-0.25427195 + \frac{2(-0.95)+1}{2}(-0.08484645) + \frac{2(-0.25)^2 + 6(-0.25)+2}{6}(-0.02515715) + \frac{2(-0.25)^3 + 9(-0.25)^2 + 11(-0.25)+3}{12}(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.25427195 + (0.25)(-0.08484645) + (0.104167)(-0.02515715) + (0.065104166)(-0.01310985)\right]$$

$$= \frac{1}{0.2}\left[-0.343208273\right]$$

$$= -1.71604$$

⑤ Find the gradient of the road at the middle point of the elevation above a datum line of seven points of road which are given below:

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Solution:- we require $\left(\frac{dy}{dx}\right)_{x=900}$

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	135						
		14					
300	149		-6				
		8		24			
600	157		-18		-56		
		$\boxed{26}$		$\boxed{-26}$		70	
$\boxed{900}$	183 (y_0)		-8		20		-86
		$\boxed{18}$		$\boxed{-6}$		-16	
1200	201		-14		4		
		4		-2			
1500	205		-16				
		-12					
1800	193						

Since $x = 900$ is in the middle of the table we use Stirling's formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18+26) - \frac{1}{12}(-6-26) + \frac{1}{60}[70-16]\right]$$

$$= \frac{1}{300}\left[22 + 2.6666 + 0.9\right] = 0.085222$$

Hence the gradient of the road at the middle point is 0.084776.

⑥ Obtain the value of $f'(0.04)$ using Bessel's formula given in the table below:

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Soln:

Since $x = 0.04$ is in the middle of the table we use central difference formula and in particular Bessel's formula.

The central difference table is

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	Δ^4	$\Delta^5 y$
0.01	-3	0.1023					
			0.0024				
0.02	-2	0.1047		0.0			
			0.0024		0.0001		
0.03	-1	0.1071		0.0001		-0.0001	
			0.0025		0.0		0.0
0.04	0	0.1096 (y_0)		0.0001		-0.0001	
			0.0026		-0.0001		
0.05	1	0.1122		0.0			
			0.0026	($\Delta^2 y_0$)			
0.06	2	0.1148					

Since $u = \dfrac{x - x_0}{h} = \dfrac{x - 0.04}{0.01}$

Taking $x_0 = 0.04$ as the origin

$y_0 = 0.1096 \quad \Delta y_0 = 0.0026 \quad \Delta y_{-1} = 0.0025, \quad \Delta y_{-2} = 0.0024$

By Bessel's formula

$$y(x_0 + uh) = \frac{1}{2}(y_0 + y_1)(u - \tfrac{1}{2})\,\Delta y_0 + \frac{u(u-1)}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{u(u - \frac{1}{2}(u-1))}{6}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{48}(\Delta^4 y_{-2} + \Delta^4 y_{-1})$$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{(3u^2 - 3u + \frac{1}{2})}{6}\Delta^3 y_{-1} + \cdots\right]$$

$$y'(x_0) = \frac{1}{0.01}\left[0.0026 - \frac{1}{4}(0 + 0.0001) + \frac{1}{12}(-0.0001) + \frac{1}{24}(-0.0001)\right]$$

$$= \frac{1}{0.24}\left[24 \times 0.0026 - 0.0006 - 0.0003\right]$$

$f'(0.04) = 0.25625.$

Numerical integration by trapezoidal and Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules, Romberg's method

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

This is known as the trapezoidal rule.

Simpson's one third rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

This is known as the Simpson's one-third rule (or) simply Simpson's rule and is most commonly used.

Note:
While applying (3), the given interval must be divided into even number of equal sub-intervals, since we find the area of two strips at a time.

Simpson's three-eight rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

which is known as Simpson's three-eight rule.

Note: While applying (4) the number of sub-intervals should be taken multiple of 3.

Formula.

AT A GLANCE				
Rule	Degree of $y(x)$	No. of intervals	Error	order
Trapezoidal rule	one	any	$\lvert E\rvert < \frac{(b-a)h^2}{12} M$	h^2
Simpson's 1/3 rule	TWO	even	$\lvert E\rvert < \frac{(b-a)h^4}{180} M$	h^4
Simpson's 3/8 rule	Three	multiple of 3	$\lvert E\rvert = \frac{3}{8}h^5$	

Problems:-

① using trapezoidal rule evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Soln:-

Here $y(x) = \frac{1}{1+x^2}$

length of the interval $= 2$

So we divide 8 equal intervals with $h = \frac{2}{8} = 0.25$

we form a table

x :	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y :	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

Trapezoidal rule,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[\text{Sum of the first and last ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$

$$= \frac{0.25}{2}\left[(0.5+0.5)+2(0.64+0.8+0.9412+0.9412+0.8+0.64)\right]$$

$$= \frac{0.25}{2}\left[1+2(5.7624)\right]$$

$$= \frac{0.25}{2}\left[12.5248\right]$$

$$= 1.5656$$

② Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h = \frac{1}{6}$ by trapezoidal rule

Soln:-

Here $y(x) = \frac{1}{1+x^2}$, $h = 1/6$

x	0	$\frac{1}{6}$	$2/6$	$3/6$	$4/6$	$5/6$	1
y	1	$\frac{36}{37}$	$9/10$	$4/5$	$9/13$	$\frac{36}{61}$	$1/2$

By Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\Big[(\text{sum of the first and last oridinate}) + 2(\text{sum of the remaining ordinates})\Big]$$

$$= \frac{(1/6)}{2}\left[\left(1+1/2\right) + 2\left(\frac{36}{37} + 9/10 + 4/5 + 9/13 + \frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[3/2 + 2[3.9554]\right]$$

$$= \frac{1}{12}\left[3/2 + 7.9108\right]$$

$$= 0.7842.$$

③ Evaluate the integral $\int_1^2 \frac{dx}{1+x^3}$ using Trapezoidal rule with two sub intervals.

Soln:-

Here $y(x) = \frac{1}{1+x^2}$, $h = 1/2 = 0.5$

x	1	1.5	2
y	0.5	0.3077	0.2

By Trapezoidal rule

$$\int_1^2 \frac{dx}{1+x^3} = \frac{h}{2}\Big[\text{sum of the first and last oridinates} + 2(\text{sum of the remaining ordinates})\Big]$$

$$= \frac{0.5}{2}\left[0.5 + 0.2 + 2(0.3077)\right]$$

$$= \frac{0.5}{2}\left[0.7 + 0.6154\right]$$

$$= \frac{0.5}{2}\left[1.3154\right] = 0.3289$$

④ dividing the rang into 10 equal parts, find the value of $\int_0^{\pi/2} \sin x \, dx$ by (i) Trapezoidal rule

(ii) simpson's rule

Soln:

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
$y=\sin x$	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

(i) By Trapezoidal rule

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{2}\left[y_0 + y_{11} + 2(y_1 + y_2 + \cdots + y_{10})\right]$$

$$h = \frac{\pi/2}{10} = \pi/20$$

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{40}(12.7062) = 0.9980$$

(ii) By Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi/2} \sin x \, dx = (h/3)\left[(y_0 + y_{11}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})\right]$$

$$= \left(\frac{\pi/20}{3}\right)\left[(0+1) + 4(3.1962) + 2(2.6569)\right]$$

$$= \frac{\pi}{60}\left[1 + 12.7848 + 5.3138\right]$$

$$= \frac{\pi}{60}\left[19.0986\right]$$

$$= 1.0000$$

⑤ using simpson's one third rule evaluate $\int_0^1 x e^x \, dx$ taking 4 intervals. Compare your result with acutual value.

Soln:

x	0	0.25	0.5	0.75	1
$y = xe^x$	0	0.321	0.824	1.588	2.718

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}[\text{Sum of the first and last even ordinates} + 2(\text{sum of remaining even ordinates}) + 4(\text{sum of odd ordinates})]$$

$$= \frac{0.25}{3}[(0+2.718) + 2(0.321+1.588) + 4(0.824)]$$

$$= \frac{0.25}{3}[2.718 + 3.818 + 3.296]$$

$$= \frac{2.458}{3} = 0.819 = 1$$

$$\int_0^1 x e^x\,dx = \int_0^1 x\,d(e^x) = [xe^x]_0^1 - \int_0^1 e^x\,dx$$

$$= (e^1 - 0) - (e^x)_0^1$$

$$= e - [e-1] = 1$$

(6) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule also check up the results by actual integration.

Soln:-

Here $b-a = 6-0 = 6$. Divide into 6 equal parts

$h = 6/6 = 1$. Hence, the table is

x	0	1	2	3	4	5	6
$\frac{1}{1+x^2} = f(x)$	1.00	0.500	0.200	0.100	0.058824	0.038462	0.27027

There are 7 ordinates ($n=6$) we can use all the formula.

(i) By Trapezoidal rule,

$$I = \int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}[(1+0.027027) + 2(0.5+0.2+0.1+0.058824+0.038462)]$$

$$= 1.41079950$$

(ii) By Simpson's one-third rule.

$$I = \frac{1}{3}[(1+0.027027) + 2(0.2+0.058824) + 4(0.5+0.1+0.038462)]$$

$= \frac{1}{3}\,(1.027027 + 0.517648 + 2.553848)$

$= 1.36617433$

(iii) By Simpson's 3/8 Rule.

$$I = \frac{3 \times 1}{8}\left[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.038462) + 2(0.1) \right]$$

$= 1.35708188$

(iv) By actual integration

$$I = \int_0^6 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^6 = \tan^{-1} 6 = 1.40564765$$

Conclusion:- Here the value by trapezoidal rule is closer to the actual value than the value by Simpson's rule.

Romberg's method

① Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Solution:-

Let $y = \frac{1}{x^2+4}$ and Let $I = \int_0^2 \frac{dx}{x^2+4}$

Take $h = 1$

The tabulated value of y are

x	0	1	2
y	0.25	0.20	0.125

using trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1) \right]$$

$= (0.5)\left[(0.25 + 0.125) + 2(0.20) \right]$

$= 0.3875$

Take $h=0.5$ The tabulated values of y are

x	0	0.5	1.0	1.5	2.0
y	0.25	0.2353	0.20	0.160	0.125

using Trapezoidal rule

$$I_2 = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$=(0.25)\left[(0.25+0.125)+2(0.2353+0.2+0.16)\right]$$

$$= 0.3914$$

Take $h=0.25$ The tabulated values of y are

x	0	0.50	0.75	1.0	1.25	1.50	1.75	2.00
y	0.25	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125

By trapezoidal rule

$$I_3 = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\dots+y_7)\right]$$

$$=\left(\frac{0.25}{2}\right)\left[(0.25+0.125)+2(0.2462+0.2353+0.2192+0.20+0.1798+0.16+0.1416)\right]$$

$$=(0.125)\left[3.1392\right]$$

$$I_3 = 0.3924$$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2-I_1}{3}\right)$$

$$= 0.3914 + \left(\frac{0.3914-0.3875}{3}\right)$$

$$I = 0.3953 \quad \text{——} (1)$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3-I_2}{3}\right)$$

$$= 0.3924 + \frac{(0.3924-0.3914)}{3}$$

$$= 0.3927 \quad \text{——} (2)$$

Since (1) and (2) are not equal we go for one more application of Trapezoidal rule taking $h = 0.125$.

Take $h = 0.125$ The tabulated values are

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875
y	0.25	0.249	0.2462	0.2415	0.2353	0.2278	0.2192	0.2098

1.00	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
0.20	0.1899	0.1798	0.1698	0.160	0.1506	0.1416	0.1331	0.125

By Trapezoidal rule

$$I_4 = \frac{h}{2}\left[(y_0 + y_{16}) + 2(y_1 + y_2 + \cdots + y_{15})\right]$$

$$= \left(\frac{0.125}{2}\right)\left[(0.25 + 0.125) + 2(0.249 + 0.2462 + \cdots + 0.1331)\right]$$

$$I_4 = 0.3926$$

Using Romberg's formula for I_3 and I_4 we have

$$I = I_4 + \left(\frac{I_4 - I_3}{3}\right)$$

$$= 0.3926 + \left(\frac{0.3926 - 0.3924}{3}\right)$$

$$I = 0.3927 \quad \text{——} (3)$$

Since (2) and (3) are almost equal we can take

$$I = \int_0^2 \frac{dx}{x^2+4} = 0.3927 \quad \text{——} (4)$$

By actual integration

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^2 \frac{dx}{x^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big]_0^2$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8} \quad \text{——} (5)$$

$\therefore$ From (4) and (5) we get $\pi/8 = 0.3927$

$\therefore \pi \approx 3.1416.$

② Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method correct to 4 decimal places. Hence deduce an approximate value of π.

Solution:

Let $y = \frac{1}{1+x^2}$ and let $I = \int_0^1 \frac{dx}{1+x^2}$

Take $h=0.5$ The tabulated values of y are

x	0	0.5	1
$y=\frac{1}{1+x^2}$	1	0.8	0.5

using Trapezoidal rule

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_2)+2y_1\right]$$

$$= \frac{0.5}{2}\left[(1+0.5)+1.6\right]$$

$$= 0.775$$

Take $h=0.25$ The tabulated values of y are

x	0	0.25	0.50	0.75	1.00
$y=\frac{1}{1+x^2}$	1	0.9412	0.80	0.64	0.5

using Trapezoidal rule,

$$I_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[(1+0.5)+2(0.9412+0.80+0.64)\right]$$

$$= 0.7828$$

Take $h=0.125$ The tabulated values of y are

x	0	0.125	0.25	0.375	0.50	0.625	0.750	0.875	1.0
$y=\frac{1}{1+x^2}$	1	0.9846	0.9412	0.8767	0.80	0.7191	0.64	0.5664	0.5

using Trapezoidal rule

$$I_3 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)\right]$$

$$= \left(\frac{0.125}{2}\right)\left[(1+0.5)+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)\right]$$

$= (0.0625)[1.5 + 2(5.528)]$

$= 0.78475$

using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right)$$

$$= 0.7828 + 0.0026$$

$$= 0.7854$$

using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.78475 + \left(\frac{0.78475 - 0.7828}{3}\right)$$

$$= 0.78475 + 0.00065$$

$$= 0.7854$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \quad \text{—} (1)$$

By actual evaluation of the definite integral we have

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \pi/4 \quad \text{—} (2)$$

From (1) and (2) we have $\pi/4 = 0.7854$.

Hence $\pi \approx 3.1416$.

TWO AND THREE POINT GAUSSIAN QUADRATURE FORMULAS

Two points Gaussian Quadrature - Problems

Formula:-

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

This formula is exact for polynomials upto degree 3.

1. Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$

Soln:

Given interval is -1 to 1 so we apply

$$\int_{-1}^{1} f(x)\, dx = f\left(-\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right) \text{ formula.}$$

Here $f(x) = \frac{1}{1+x^2}$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$\therefore \int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} \frac{1}{1+x^2} dx = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} = 1.5$$

But actual integration

$$\int_{-1}^{1} \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1)$$

$$= 2 \tan^{-1}(1)$$

$$= 2\, \frac{\pi}{4}$$

$$= \frac{\pi}{2} = 1.5708$$

Here the error due to two-point formula is 0.0708

2. Apply Gauss two-point formula to evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$

Soln:

Given interval is 0 to 1, to make them as -1 to 1

$$\int_{0}^{1} \frac{dx}{1+x^2} = \frac{1}{2} \int_{-1}^{1} \frac{dx}{1+x^2} \quad \left[\because \frac{1}{1+x^2} \text{ is an even function}\right]$$

$$= \frac{1}{2}[1.5] \quad [\text{by first problem}]$$

$$= 0.75$$

③. Using Gaussian two-point formula evaluate

(i) $\int_{-1}^{1} (3x^2 + 5x^4)\, dx$ (ii) $\int_{0}^{1} (3x^2+5x^4)\, dx$

Solution:-

(i) Given interval is -1 to 1

Hence we can apply the formula

$$\int_{-1}^{1} f(x)\, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = 3x^2 + 5x^4$

$$f\left(\frac{-1}{\sqrt{3}}\right) = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \frac{5}{9} = \frac{14}{9} = 1.556$$

$$f\left(\frac{+1}{\sqrt{3}}\right) = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \frac{5}{9} = \frac{14}{9} = 1.556$$

$$\int_{-1}^{1} (3x^2 + 5x^4)\, dx = (1.556 + 1.556) = 3.112$$

(ii) Given interval is 0 to 1, so to make them as -1 to 1

Soln:- $$\int_{0}^{1} (3x^2+5x^4)\, dx = \frac{1}{2}\int_{-1}^{1} (3x^2+5x^4)\, dx$$

[$\because 3x^2+5x^4$ is an even function]

$$= \frac{1}{2}[3.112] = 1.556$$

④ Evaluate $\int_{-2}^{2} e^{-x/2}\, dx$ by Gauss two point formula.

Soln:-

Given The range is not $(-1, 1)$ so by using the formula to make them as $(-1, 1)$

$$x = \frac{b-a}{2} z + \frac{b+a}{2}.$$ Here $a = -2$; $b = 2$

$$x = \frac{2+2}{2} z + \frac{2-2}{2}$$

$$x = 2z \Rightarrow z = \frac{x}{2}$$

$$dx = 2dz$$

$$\int_{-2}^{2} e^{-x/2}\,dx = \int_{-1}^{1} e^{-z}(2\,dz)$$

$$= 2\int_{-1}^{1} e^{-z}\,dz$$

$$= 2\left[f\left(\tfrac{-1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)\right]$$

Here $f(z) = e^{-z}$

$$f\left(\tfrac{-1}{\sqrt{3}}\right) = e^{1/\sqrt{3}} = 1.7813$$

$$f\left(\tfrac{1}{\sqrt{3}}\right) = e^{-1/\sqrt{3}} = 0.5614$$

$$= 2\,[0.5614 + 1.7813]$$

$$= 4.6854$$

<u>Three points Gaussian quadrature</u>

Formula $\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9} f(0)$

This formula is exact for polynomials upto degree 5.

① using Gaussian three - point formula evaluate

(i) $\int_{-1}^{1} (3x^2 + 5x^4)\,dx$

(ii) $\int_{0}^{1} (3x^2 + 5x^4)\,dx$

also compare with exact values

Soln:-

Let $f(x) = 3x^2 + 5x^4$ [Range given is exact form]

$f(0) = 0$

$$f\left(-\sqrt{3/5}\right) = 3\left(3/5\right) + 5\left(3/5\right)^2 = 9/5 + 9/5 = 18/5$$

$$f\left(\sqrt{3/5}\right) = 3\left(3/5\right) + 5\left(3/5\right)^2 = 9/5 + 9/5 = 18/5$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left[\frac{18}{5} + \frac{18}{5}\right] + 0$$

$$= \frac{5}{9}\,\frac{36}{5} = 4 \quad -①$$

Exact value

$$\int_{-1}^{1} (3x^2+5x^4)\,dx = 2\int_{0}^{1} (3x^2+5x^4)\,dx \quad [\because 3x^2+5x^4 \text{ is an even function}]$$

$$= 2\left[\frac{3x^3}{3} + \frac{5x^5}{5}\right]_0^1$$

$$= 2\left[x^3 + x^5\right]_0^1$$

$$= 2\left[(1+1) - (0+0)\right]$$

$$= 4$$

we get exact value by using Gaussian three-point formula

(ii) $\int_{0}^{1} (3x^2+5x^4)\,dx$ [The range is not exact form]

$$\int_{0}^{1} (3x^2+5x^4)\,dx = \frac{1}{2}\int_{-1}^{1} (3x^2+5x^4)\,dx \quad [\because 3x^2+5x^4 \text{ is an even function}]$$

$$= \frac{1}{2}[4] = 2 \quad [\text{by } (1)]$$

② using three-point Gaussian quadrature formula, evalute

(i) $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

(ii) $\int_{0}^{1} \frac{1}{1+t^2}\,dt$

Soln:

Let $f(x) = \frac{1}{1+x^2}$ [Range given is exact form]

$$f(0) = \frac{1}{1+0} = 1$$

$$f\left(-\sqrt{3/5}\right) = \frac{1}{1+3/5} = \frac{1}{8/5} = 5/8$$

$$f\left(-\sqrt{3/5}\right) = \frac{1}{1+3/5} = \frac{1}{8/5} = 5/8$$

Three-point Gaussian quadrature formula is

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 5/9\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + 8/9\, f(0)$$

$$= 5/9\left[5/8 + 5/8\right] + 8/9\,(1)$$

$$= 5/9\left[2(5/8)\right] + 8/9$$

$$= \frac{50}{72} + 8/9 = \frac{19}{12} = 1.5833 \quad -①$$

Actual value

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2\int_{0}^{1} \frac{1}{1+x^2}\,dx \quad \left[\frac{1}{1+x^2} \text{ is an even formula}\right]$$

$$= 2\left[\tan^{-1} x\right]_0^1$$

$$= 2\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= 2\left[\pi/4\right]$$

$$= \pi/2 = 1.5708$$

(ii) Range given is not exact form

$$\therefore \int_{0}^{1} \frac{1}{1+t^2}\,dt = 1/2 \int_{-1}^{1} \frac{1}{1+t^2}\,dt \quad \left[\because \frac{1}{1+t^2} \text{ is an even function}\right]$$

$$= 1/2\left[1.5833\right] = 0.79165$$

③ Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian three point formula

Soln:-

Let $f(x) = \frac{x^2+2x+1}{1+(x+1)^4}$ [Range given is not ~~for~~ exact form]

Let $x = \frac{b-a}{2} z + \frac{b+a}{2}$ $\quad [a=0, b=2]$

$= \frac{2-0}{2} z + \frac{2+0}{2}$

$x = z+1$ | $x = 0 \Rightarrow z = -1$

$dx = dz$ | $x = 2 \Rightarrow z = 1$

$$\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4} dx = \int_{-1}^{1} \frac{(z+1)^2 + 2(z+1)+1}{1+[(z+1)+1]^4} dz$$

$$= \int_{-1}^{1} \frac{z^2+2z+1+2z+2+1}{1+(z+2)^4} dz$$

$$= \int_{-1}^{1} \frac{z^2+4z+4}{(z+2)^4+1} dz \quad \text{---} \; (1)$$

[Range given is in exact form]

$$\therefore f(z) = \frac{z^2+4z+4}{(z+2)^4+1}$$

$$f(z) = \frac{(z+2)^2}{(z+2)^4+1}$$

$$f(0) = \frac{2^2}{2^4+1} = \frac{4}{17}$$

$$f\left[-\sqrt{3/5}\right] = \frac{\left[-\sqrt{3/5}+2\right]^2}{\left[-\sqrt{3/5}+2\right]^4+1} = \frac{1.50161}{3.2548} = 0.4614$$

$$f\left[+\sqrt{3/5}\right] = \frac{\left[\sqrt{3/5}+2\right]^2}{\left[\sqrt{3/5}+2\right]^4+1} = \frac{7.69839}{60.2652} = 0.12774$$

$$\therefore (1) \Rightarrow \int_{-1}^{1} f(z)\, dz = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left[0.4614 + 0.12774\right] + \frac{8}{9}\left[\frac{4}{17}\right]$$

$$= 0.3273 + 0.2092 = 0.5365$$

$$\therefore (1) \Rightarrow \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4} dx = \int_{-1}^{1} \frac{z^2+4z+4}{(z+2)^2+1} dz = 0.5365$$

DOUBLE INTEGRALS USING TRAPEZOIDAL AND SIMPSON'S RULES

Trapezoidal rule

$$I = \frac{hk}{4}\left[\text{sum of the values of } f(x,y) \text{ at the four corner points}\right]$$

Simpson's rule for double integration

$$I = \frac{hk}{9}\Big[(\text{sum of the values of } f \text{ at the four corners})$$
$$+ 2(\text{sum of the values of } f \text{ at the odd positions on the boundary except the corners})$$
$$+ 4(\text{sum of the values of } f \text{ at the even positions on the boundary})$$
$$+ \{4(\text{sum of the values of } f \text{ at odd poistions})$$
$$8(\text{sum of the values of } f \text{ at even poistions}) \text{ on the odd rows of the matrix except boundary rows}\}$$
$$+ 8\{(\text{sum of the values of } f \text{ at the odd poisitions})$$
$$+16(\text{sum of the values of } f \text{ at the even poisitions}) \text{ on the even rows of the matrix})\}\Big]$$

problems:-

① Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoidal and Simpson's rule verify your result by actual integration.

Solution:-

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1 \quad \text{and} \quad k = \frac{1.4-1}{4} = 0.1$$

Get the values of $f(x,y) = \frac{1}{xy}$ at nodal points.

y/x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Case (i): By Trapezoidal Rule, we get

$I = \frac{hk}{4}$ [(Sum of values of f at the four corners + 2 (Sum of values of f at the remaining nodes on the boundary) + 4 (Sum of the values of f at the interior nodes)]

$$= \frac{(0.1)(0.1)}{4}\left[(0.5) + 0.4167 + 0.3571 + 0.2976\right]$$

$$+2\left[0.3846 + 0.4167 + 0.4545 + 0.4762 + 0.4545 + 0.4348 + 0.3788 + 0.3472 + 0.3205 + 0.3106 + 0.3247 + 0.3401\right]$$

$$+4\left[0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 + 0.3623 + 0.3663 + 0.3497 + 0.3344\right]$$

$$= \frac{0.01}{4}\left[1.5714 + 9.2864 + 13.7188\right]$$

$$= 0.0614$$

Case (ii): By Simpson's rule

$I = \frac{hk}{9}$ [(sum of the values of f at the four corners)

+2 (sum of the values of f at the odd positions on the boundary except the corners)

+4 (sum of the values of f at the even positions on the boundary)

+ {4 (sum of the values of f at odd positions)

+ 8 (sum of the values of f at even positions) on the odd row of the matrix except boundary rows}

+ {8 (sum of values of f at the odd positions)

+16 (sum of values of f at the even positions) on the even rows of the matrix}]

$$= \frac{(0.1)(0.1)}{9}\Big[(0.5 + 0.4167 + 0.3571 + 0.2976)$$

$$+ 2(0.4167 + 0.4545 + 0.3472 + 0.3247)$$

$$+ 4(0.3846 + 0.4545 + 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 + 0.3401)$$

$$+ 4(0.3788)$$

$$+ 8(0.3968 + 0.3623)$$

$$+ 8(0.3497 + 0.4132)$$

$$+ 16(0.3663 + 0.3344 + 0.4329 + 0.3953)\Big]$$

$$= \frac{0.01}{9}[55.2116] = 0.0657$$

Case (3) :- By actual integration

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \left(\int_1^{1.4} \frac{1}{y}\,dy\right)\left(\int_2^{2.4} \frac{1}{x}\,dx\right)$$

$$= (\log y)_1^{1.4} (\log x)_2^{2.4}$$

$$= (\log 1.4)[\log 2.4 - \log 2]$$

$$= \log(1.4)\,\log(1.2)$$

$$= 0.0613.$$

we get the actual value and the value by Simpson's rule are equal while the value by trapezoidal rule differs only by 0.0001.

② Evaluate $\int_0^2\int_0^2 f(x, y)\,dx\,dy$ by Trapezoidal rule for the following data.

y/x	0	0.5	1	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

Solution:-

Here $h = 0.5$

$k = 1$

$$I = \int_0^2 \int_0^2 f(x, y)\, dx\, dy$$

$$I = \frac{hk}{4}\Big[(\text{sum of values of } f \text{ at the four corners})$$

$$+ 2\,(\text{sum of the values of } f \text{ at the remaining nodes on the boundary})$$

$$+ 4\,(\text{sum of the values of } f \text{ at the interior nodes})\Big]$$

$$= \frac{(0.5)(1)}{4}\left[(2+5+14+4) + 2(3+3+4+5+11+11+8+6) + 4(4+6+9)\right]$$

$$= \frac{(0.5)(1)}{4}\left[25 + 2(51) + 4(19)\right]$$

$$= (0.125)[203]$$

$$= 25.375$$

③ Using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1 \int_0^1 \frac{1}{1+x+y}\, dx\, dy$ taking $h = k = 0.5$

Soln:-

y/x	0	0.5	1
0	1	0.6667	0.5
0.5	0.6667	0.5	0.4
1	0.5	0.4	0.3333

Simpson's rule:-

$$I = \frac{hk}{9}\Big[(\text{sum of the values of } f \text{ at the four corners})$$

$$+ 2\,(\text{sum of the values of } f \text{ at the odd positions on the boundary except the corners})$$

$$+ 4\,(\text{sum of the values of } f \text{ at the even positions on the boundary})$$

$$+ \{4\,(\text{sum of the values of } f \text{ at odd positions})$$

$$+ 8\,(\text{sum of the values of } f \text{ at even positions}) \text{ on the odd row of the matrix except boundary rows}\}$$

$+\{8$ (sum of the values of f at the odd positions)

$+16$ (sum of the values of f at the even positions) on the even rows of the matrix$\}]$

$$I = \frac{(0.5)(0.5)}{9}\Big[(1+0.5+0.3333+0.5)+2(0)+4(0.6667+0.6667+0.4+0.4)+\{4(0)+8(0)\}+\{8(0)+16(0.5)\}\Big]$$

$$= (0.02778)[(2.3333)+4(2.1334)+8]$$

$$= (0.02778)(18.8669)$$

$$= 0.5241$$

④ Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x^2+y^2}$ numerically with $h=0.2$ along x-direction and $k=0.25$ along y-direction.

Solution:-

y/x	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

By Trapezoidal rule

$\int_1^2\int_1^2 \frac{1}{x^2+y^2}\,dx\,dy = \frac{hk}{4}$[sum of values of f at the four corners

+ 2 (sum of the values of f at the remaining nodes on the boundary)

+ 4 (sum of the values of f at the interior nodes)]

$$= \frac{(0.2)\ (0.25)}{4} [(0.5 + 0.2 + 0.125 + 0.2) + 2(0.2462 + 0.3077$$
$$+ 0.3902 + 0.4098 + 0.3378 + 0.2809 + 0.2359 + 0.1798 + 0.16$$
$$+ 0.1416 + 0.1381 + 0.1524 + 0.1679 + 0.1838)$$
$$+ 4(0.3331 + 0.2839 + 0.2426 + 0.2082 + 0.2710 + 0.2375$$
$$+ 0.2079 + 0.1821 + 0.2221 + 0.1991 + 0.1779 + 0.1587$$
$$+ 0.1838 + 0.1679 + 0.1524 + 0.1381)]$$

$$= (0.0125) [1.025 + 6.6566 + 13.4652]$$

$$= 0.2643$$

UNIT - IV

Initial value problems for ordinary differential equations

Taylor series method:

$$y_{n+1} = y_n + \frac{(x-x_0)}{1!} y_n' + \frac{(x-x_0)^2}{2!} y_n'' + \frac{(x-x_0)^3}{3!} y_n''' + \cdots$$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{where } h = (x - x_0)$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Problems:

① Solve $y' = x + y$; $y(0) = 1$ by Taylor's series method. Find the value of y at $x = 0.1$ and $x = 0.2$

Soln:

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

Here, $x_0 = 0$, $y_0 = 1$

$y' = x + y$, $\quad y_0' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y'$ $\quad y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$ $\quad y_0''' = y_0'' = 2$

$$\therefore y = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + (x-0)(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(2) + \cdots$$

$$y = 1 + x + \frac{x^2}{2}(2) + \frac{x^3}{6}(2) + \cdots$$

$$y = 1 + x + x^2 + x^3/3 + \cdots$$

$$\therefore y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} = 1.1103$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} = 1.2426$$

Another method

$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$ since $h = x_1 - x_0 = 0.1 - 0 = 0$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$y(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \cdots$$

$$= 1.1103$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$y' = x + y$

$y_1' = x_1 + y_1 = 0.1 + 1.1103 = 1.2103$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

$$\therefore y(0.2) = 1.1103 + \frac{(0.1)}{1!}(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103)$$

$y(0.2) = 1.24275$//

2. Solve $\frac{dy}{dx} = y^2 + x^2$, $y(0) = 1$ by Taylor series method Find y at $x = 0.1, 0.2, 0.3, 0.4$

Soln:

Taylor series formula

$$y_1 = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

given, $x_0 = 0, \quad y_0 = 1$

$y' = y^2 + x^2$

$y_0' = y_0^2 + x_0^2 = 1 + 0 = 1$

$y'' = 2yy' + 2x$

$y''' = 2[yy'' + y'y' + 1]$

$y_0'' = 2y_0 y_0' + 2x_0 = 2(1)(1) + 2(0) = 2$

$y_0''' = 2[y_0 y_0'' + y_0'^2] + \boxed{2} = 2[(1)(2) + (1)^2] + \boxed{2}$

$= 2[2 + 1] + \boxed{2}$

$= 6 + 2$

$= 8$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{6}(8) + \cdots$$

$$y = 1 + x + x^2 + \frac{4x^3}{3} + \cdots$$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \tfrac{4}{3}(0.1)^3 + \cdots = 1.1113$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \tfrac{4}{3}(0.2)^3 + \cdots = 1.25066$$

$$y(0.3) = 1 + (0.3) + (0.3)^2 + \tfrac{4}{3}(0.3)^3 + \cdots = 1.426$$

$$y(0.4) = 1 + (0.4) + (0.4)^2 + \tfrac{4}{3}(0.4)^3 + \cdots = 1.64533$$

Home work:-

1. using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$.
2. using Taylor series method find $y(1,1)$ given $y' = x + y$, $y(1) = 0$.
3. using Taylor series method with the first five terms in the expansion find $y(0.1)$ correct to three decimal places given that $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$

Soln:-

$$\frac{dy}{dx} = y' = e^x - y^2 \qquad x_0 = 0,\ y_0 = 1$$

$$y' = e^x - y^2 \qquad y_0' = e^{x_0} - y_0^2 = 1 - 1 = 0$$

$$y'' = e^x - 2yy' \qquad y_0'' = e^{x_0} - 2y_0y_0' = 1 - 2(0) = 1$$

$$y''' = e^x - 2[yy'' + y'y'] \qquad y_0''' = e^{x_0} - 2[y_0y_0'' + (y_0')^2]$$

$$= 1 - 2[1(1) + 0] = 1 - 2 = -1$$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$= 1 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-1)$$

$$= 1 + 0 + \frac{x^2}{2} + \frac{x^3}{6}(-1)$$

$$y = 1 + \frac{x^2}{2} - \frac{x^3}{6}$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3!} = 1.004833 = 1.005$$

④ By means of Taylor series expansion, find y at $x = 0.1, 0.2$ correct to three significant digits given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Soln:

Here $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ $\quad h = 0.1$

$y' = 2y + 3e^x \qquad y_0' = 2y_0 + 3e^{x_0} = 3$

$y'' = 2y' + 3e^x \qquad y_0'' = 2y_0' + 3e^{x_0} = 9$

$y''' = 2y'' + 3e^x \qquad y_0''' = 18 + 3 = 21$

$y^{iv} = 2y''' + 3e^x \qquad y_0^{iv} = 42 + 3 = 45$

$$\therefore y_1 = y_0 + \frac{h}{1} y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{iv} + \cdots$$

$$y(0.1) = y_1 = 0 + (0.1)(3) + \frac{(0.01)}{2}(9) + \frac{(0.001)}{6}(21) + \frac{(0.0001)}{24}(45) + \cdots$$

$$= 0.3 + 0.045 + 0.0035 + 0.0001875 + \cdots$$

$$= 0.3486875$$

$$= 0.349 \text{ (three decimals)}$$

$y_1' = 2y_1 + 3e^{x_1} = 0.3486875 \times 2 + 3e^{0.1} = 4.012887$

$y_1'' = 2y_1' + 3e^{x_1} = 11.025774$

$y_1''' = 2y_1'' + 3e^{x_1} = 25.3670608.$

$$y_2 = y(0.2) = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2} y_1'' + \cdots$$

$$= 0.3486875 + (0.1)(4.012887) + \frac{0.01}{2}(11.341286) + \frac{(0.001)}{6}(25.99808) + \cdots$$

$$= 0.8110156 = 0.811 \text{ (correct to three decimal places)}$$

The exact value of $y(0.1) = 0.3486955$ and $y(0.2) = 0.8112658$

<u>Taylor series method for simultaneous first order differential equations</u>

① Solve the system of equations $\frac{dy}{dx} = z - x^2$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$ to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

Soln:-

$x_0 = 0 \quad y_0 = 1 \quad z_0 = 1$

$y' = z - x^2 \qquad y_0' = z_0 - x_0^2$

$= 1 - 0 = 1$

$y'' = z' - 2x \qquad y_0'' = z_0' - 2x_0$

$= 1 - 2(0) = 1$

$y''' = z'' - 2(1) \qquad y_0''' = z_0'' - 2$

$= 2 - 2 = 0$

$z' = y + x \qquad z_0' = y_0 + x_0$

$= 1 + 0 = 1$

$z'' = y' + 1 \qquad z_0'' = y_0' + 1 = 1 + 1 = 2$

$z''' = y'' \qquad z_0''' = y_0'' = 1$

$$\therefore y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$y = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2}(1) + \frac{(x-0)^3}{3!}(0) + \cdots$$

$$y = 1 + x + x^2/2$$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$$

$$z = z_0 + \frac{(x-x_0)}{1!} z_0' + \frac{(x-x_0)^2}{2!} z_0'' + \frac{(x-x_0)^3}{3!} z_0''' + \cdots$$

$$z = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(1) \cdots$$

$$z(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{6}$$

$$z(0.1) = 1.110166 \text{ // Ans}$$

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

① By Taylor's series method find $y(0.1)$ given that $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$

Soln:- Here $x_0 = 0 \quad y_0 = 1 \quad y_0' = 0$

Given $y'' = y + xy'$

$y''' = y' + xy'' + y'$

$= 2y' + xy''$

$y_0'' = y_0 + x_0 y_0' = 1 + (0)(0) = 1$

$y_0''' = 2y_0' + x_0 y_0'' = 2(0) + (0)(1) = 0$

$y_0^{iv} = 3y_0'' + x_0 y_0''' = 3(1) + (0)(0)$

$= 3$

$$y^{iv} = 2y'' + xy''' + y''$$

$$= 3y'' + xy'''$$

$$\therefore y(x) = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y^{iv}$$

$$= 1 + 0 + \frac{x^2}{2}(1) + 0 + \frac{x^4}{24}(3) + \cdots$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8}$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8}$$

$$= 1.0050$$

② Evaluate the values of $y(0.1)$ and $y(0.2)$ given $y'' - x(y')^2 + y^2 = 0$ $y(0) = 1$, $y'(0) = 0$ by using Taylor series method.

Soln:-

Given $y'' = x(y')^2 - y^2$, $x_0 = 0$ $y_0 = 1$ $y_0' = 0$

$$y''' = x2y'y'' + (y')^2 - 2yy'$$

$$= 2xy'y'' + (y')^2 - 2yy'$$

$$= y'[2xy'' + y' - 2y]$$

$$y_0'' = x_0(y_0')^2 - y_0^2$$

$$= (0)(0)^2 - (1)^2 = -1$$

$$y_0''' = 2x_0y_0'y_0'' + (y_0')^2 - 2y_0y_0'$$

$$= 2(0)(0)(-1) + (0)^2 - 2(1)(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$y^{iv} = y'[2(xy''' + y'') + y'' - 2y'] + (2xy'' + y' - 2y)y''$$

$$= 2xy'y''' + 2y'y'' - 2(y')^2 + 2x(y'')^2 + y'y'' - 2yy'$$

$$= 2xy'y''' + 4y'y'' + 2x(y'')^2 - 2(y')^2 - 2yy''$$

$$y_0^{(iv)} = 2x_0y_0'y_0''' + 4y_0'y_0'' + 2x_0(y_0'')^2 - 2(y_0')^2 - 2y_0y_0''$$

$$= 0 + 0 + 0 - 2(0) - 2(1)(-1)$$

$$= 2$$

$$\therefore y(x) = y_0 + xy_0' + \frac{x^2}{2}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \cdots$$

$$= 1 + (0) + \frac{x^2}{2}(-1) + \frac{x^4}{24}(2) + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{1}{12}x^4 + \cdots$$

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{12} + \cdots = 1 - 0.005 + 0.000008$$

$$= 0.995008 = 0.995$$ (correct to 3 decimal places)

Euler and modified Euler method.

$$\boxed{\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ y_2 &= y_1 + h f(x_1, y_1) \\ &\vdots \\ y_{n+1} &= y_n + h f(x_n, y_n), \quad n = 0, 1, 2 \ldots \end{aligned}}$$

① using Euler's method find $y(0.2)$ and $y(0.4)$, $y(0.6)$ from $\frac{dy}{dx} = x + y$, $y(0) = 1$ with $h = 0.2$

Soln:

$f(x, y) = x + y$, $\quad x_0 = 0$, $y_0 = 1$ $\quad x_1 = 0.2$, $x_2 = 0.4$

By Euler algorithm,

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.2)[x_0 + y_0]$$

$$= 1 + (0.2)(0 + 1)$$

$$y_1 = 1.2$$

ie) $y(0.2) = 1.2$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 1.2 + 0.2[x_1 + y_1]$$

$$= 1.2 + (0.2)[0.2 + 1.2]$$

$$= 1.2 + (0.2)[1.4]$$

$$= 1.2 + 0.28$$

$$y_2 = 1.48$$

$$y(0.4) = 1.48$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 1.48 + (0.2)[x_2 + y_2]$$

$$= 1.48 + (0.2)[0.4 + 1.48]$$

$$= 1.48 + 0.376$$

$$y(0.6) = 1.856$$

② Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem. $\frac{dy}{dx} = \frac{1}{2}(x^2+1)y^2$, $y(0.2) = 1.1114$.

Soln:

Given $f(x,y) = \frac{1}{2}(x^2+1)y^2$, $\quad x_0 = 0.2$, $y_0 = 1.1114$, $x_1 = 0.3$

By Euler,

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= y_0 + h[\frac{1}{2}(x_0^2+1)y_0^2]$$

$$= 1.1114 + \frac{(0.1)}{2}[((0.2)^2+1)(1.1114)^2]$$

$$= 1.1114 + \frac{0.1}{2}[1.2846184]$$

$$= 1.1114 + 0.0642$$

$$y(0.3) = 1.1756$$

③ Compute y at $x = 0.25$ by modified Euler method given $y' = 2xy$, $y(0)=1$.

Soln:

Given $f(x,y) = 2xy$, $x_0 = 0$, $y_0 = 1$, $h = 0.25$, $x_1 = 0.25$

By modified Euler method,

$$y_{n+1} = y_n + h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(x_n, y_n))$$

$$y_1 = y_0 + h f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}(2x_0 y_0)\right)$$

$$= 1 + (0.25)[f(0.125, 1 + (0.25)(0)(1)]$$

$$= 1 + 0.25[2(0.125)(1)]$$

$$= 1 + 0.00625$$

$$= 1.00625$$

④ Using modified Euler's method, compute $y(0.1)$ with $h=0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$.

$x_1 = 0.1$ $\quad h = 0.1$

Soln: Given $f(x,y) = y - \frac{2x}{y}$, $x_0 = 0$ $y_0 = 1$

By modified Euler method.

$$y_1 = y_0 + h[f[x_0 + h/2, y_0 + \frac{h}{2} f(x_0, y_0)]$$

$$= 1 + (0.1)[f(0 + \frac{0.1}{2}), 1 + \frac{0.1}{2}[y_0 - \frac{2x_0}{y_0}]]$$

$= 1 + (0.1)\left[f\left[(0.05), 1 + (0.05)\,[1-0]\right]\right]$

$= 1 + (0.1)\left[1.05 - \frac{2(0.05)}{1.05} \right]$

$= 1 + (0.1)\,[1.05 - 0.0952]$

$= 1 + (0.1)\,[0.9548]$

$= 1 + 0.09548$

$= 1.09548$ //Ans

⑤ using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 1$

Soln:

Given $f(x, y) = y - x^2 + 1$, $x_0 = 0$, $y_0 = 1$ $h = 0.1$, $x_1 = 0.1$

By modified Euler method.

$y_{n+1} = y_n + h\, f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$

$y_1 = y_0 + h f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$

$= 0.5 + (0.2)\, f\left[0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2} f(0, 0.5)\right]$

$= 0.5 + (0.2)\, f\left[0.1, 0.5 + (0.1)\,[0.5 - 0 + 1]\right]$

$= 0.5 + (0.2)\, f\,[0.1, 0.5 + (0.1)\,[1.5]]$

$= 0.5 + (0.2)\, f\,[0.1, 0.5 + 0.15]$

$= 0.5 + (0.2)\, f\,[0.1, 0.65]$

$= 0.5 + (0.2)\, f\,[0.1, 0.65]$

$= 0.5 + (0.2)\,[0.65 - (0.1)^2 + 1]$

$= 0.5 + (0.2)\,[0.65 - 0.01 + 1]$

$= 0.5 + (0.2)\,[1.64]$

$= 0.5 + 0.328$

$y_1 = 0.828$ //Ans.

RUNGE - KUTTA METHOD : (FOURTH ORDER)

$$y_1 = y_0 + \Delta y$$

where $K_1 = h f(x_0, y_0)$

$$K_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h f\left(x_0 + h,\ y_0 + K_3\right)$$

and $\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

① Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge-kutta method of fourth order.

Soln:

Given $y' = x^3 + y = f(x, y)$, $x_0 = 0$. $y_0 = 2$, $x_1 = 0.2$

Fourth order RK algorithm

$$K_1 = h f(x, y)$$

$$K_2 = h f\left[x + h/2,\ y + \frac{K_1}{2}\right]$$

$$K_3 = h f\left[x + \frac{h}{2},\ y + \frac{K_2}{2}\right]$$

$$K_4 = h f\left[x + h,\ y + K_3\right]$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(x+h) = y(x) + \Delta y$$

First method

$$K_1 = h f(x_0, y_0)$$

$$= 0.2\left[x_0^3 + y_0\right]$$

$$= (0.2)\left[0 + 2\right]$$

$$= 0.4$$

$$K_2 = h f\left[x_0 + h/2,\ y_0 + \frac{K_1}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2},\ 2 + \frac{0.4}{2}\right]$$

$$= (0.2) f\left[0 + 0.1,\ 2 + 0.2\right]$$

$$= (0.2) f\left[0.1,\ 2.2\right]$$

$= (0.2)[(0.1)^3 + 2.2]$

$= (0.2)[2.201]$

$= 0.4402.$

$$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$$

$= (0.2) f\left[0 + \frac{0.2}{2}, 2 + \frac{0.4402}{2}\right]$

$= (0.2) f[0.1, 2.2201]$

$= (0.2)[(0.1)^3 + 2.2201]$

$= (0.2)[2.2211]$

$= 0.44422$

$$K_4 = hf[x_0 + h, y_0 + K_3]$$

$= (0.2) f[0 + 0.2, 2 + 0.44422]$

$= (0.2) f[0.2, 2.44422]$

$= (0.2)[(0.2)^3 + 2.44422]$

$= (0.2)[2.45222]$

$= 0.490444$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$= \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444]$

$= \frac{1}{6}[2.65928]$

$\Delta y = 0.44321$

$y(0.2) = y_1 = y_0 + \Delta y = 2 + 0.44321 = 2.44321 = 2.443$

(Correct to 3 decimals)

Again apply R.K method (Second integral)

$K_1 = hf(x_1, y_1) = (0.2) f[0.2, 2.443]$

$= (0.2)[(0.2)^3 + 2.443]$

$= (0.2)\ [2.451]$

$K_1 = 0.4902$

$K_2 = hf\left[x_1 + h/2,\ y_1 + \frac{K_1}{2}\right]$

$= (0.2)\ f\left[0.2 + \frac{0.2}{2},\ 2.443 + \frac{0.4902}{2}\right]$

$= (0.2)\ f\ [0.3,\ 2.6881]$

$= (0.2)\ [(0.3)^3 + 2.6881]$

$= (0.2)\ [2.7151]$

$K_2 = 0.5430$

$K_3 = hf\left[x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2}\right]$

$= (0.2)\ f\left[0.2 + \frac{0.2}{2},\ 2.443 + \frac{0.543}{2}\right]$

$= (0.2)\ f\ [0.3,\ 2.7145]$

$= (0.2)\ [(0.3)^3 + 2.7145]$

$= (0.2)\ [2.7415]$

$= 0.5483$

$K_4 = hf\ [x_1 + h,\ y_1 + K_3]$

$= (0.2)\ f\ [0.2 + 0.2,\ 2.443 + 0.5483]$

$= (0.2)\ f\ [0.4,\ 2.9913]$

$= (0.2)\ [(0.4)^3 + 2.9913]$

$= (0.2)\ (3.0553)$

$= 0.6111$

$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.4902 + 2\cdot(0.543) + 2(0.5483) + 0.6111\right]$

$= \frac{1}{6}\ [3.2839]$

$= 0.5473$

$y(0.4) = y_2 = y_1 + \Delta y = 2.443 + 0.5473 = 2.99$ (correct to 3 decimal)

Again apply R.K method (third interval)

Here $x_2 = 0.4 \quad y_2 = 2.99$

$K_1 = hf(x_2, y_2)$

$= (0.2) f(0.4, 2.99)$

$= (0.2)[(0.4)^3 + 2.99] = (0.2)[3.054] = 0.6108$

$K_2 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right]$

$= (0.2) f\left[(0.4) + \frac{0.2}{2}, 2.99 + \frac{0.6108}{2}\right]$

$= (0.2) f[0.5, 3.2954]$

$= (0.2)[(0.5)^2 + 3.2954] = (0.2)(3.4204) = 0.6841$

$K_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right]$

$= (0.2) f\left[0.4 + \frac{0.2}{2}, 2.99 + \frac{0.6841}{2}\right]$

$= (0.2) f[0.5, 3.3321]$

$= (0.2)[(0.5)^3 + 3.3321]$

$= (0.2)[3.4571] = 0.6914$

$K_4 = hf[x_2 + h, y_2 + K_3]$

$= (0.2) f[0.4 + 0.2, 2.99 + 0.6914]$

$= (0.2) f[0.6, 3.6814]$

$= (0.2)[(0.6)^3 + 3.6814]$

$= (0.2)[3.8974]$

$= 0.7795$

$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= \frac{1}{6}[0.6108 + 2(0.6841) + 2(0.6914) + 0.7795]$

$= \frac{1}{6}[4.1413]$

$\Delta y = 0.6902$

$y(0.6) = y_3 = y_2 + \Delta y = 2.99 + 0.6902 = 3.68$

x	0	0.2	0.4	0.6
y	2	2.443	2.99	3.68

② using R.K method of 4th order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$

Soln:

Given $y' = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = 0.2$

$$K_1 = h f(x_0, y_0) = (0.2)\left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right] = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$K_2 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= (0.2) f[0.1, 1.1] = (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$$

$$= (0.2)\left[\frac{1.2}{1.22}\right]$$

$$= (0.2)[0.9836]$$

$$= 0.19672$$

$$K_3 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right]$$

$$= (0.2) f[0.1, 1.0983606]$$

$$= (0.2)\left[\frac{(1.0983606)^2 - (0.1)^2}{(1.0983606)^2 + (0.1)^2}\right]$$

$$= 0.1967$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

$$= (0.2) f(0.2, 1.1967)$$

$$= (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$$

$$= 0.1891$$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19598$$

$$y(0.2) = y_1 = y_0 + \Delta y = 1 + 0.19598 = 1.19598$$

RUNGE - KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

① consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order R.K method, find $y(0.2)$

Soln:-

let $t = x$

$y'' = 2y' - 2y + e^{2x}\sin x$, $y(0) = -0.4$ $y'(0) = -0.6$ $h = 0.2$

Setting $y' = z$ the equation becomes

$$z' = 2z - 2y + e^{2x}\sin x.$$

$f_1(x,y,z) = \frac{dy}{dx} = z$, $f_2(x,y,z) = \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$

Given: $y_0 = -0.4$, $z_0 = y_0' = -0.6$, $x_0 = 0$

$K_1 = hf_1(x_0, y_0, z_0)$

$= (0.2)[z_0]$

$= (0.2)[-0.6]$

$= -0.12$

$l_1 = hf_2(x_0, y_0, z_0)$

$= (0.2)[2z_0 - 2y_0 + e^{2x_0}\sin x_0]$

$= (0.2)[2(-0.6) - 2(-0.4) + e^{2(0)}\sin(0)]$

$= (0.2)[-0.12 + 0.8]$

$\boxed{l_1 = 0.136}$

$K_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.2)f_1\left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 + \frac{0.136}{2}\right]$

$= (0.2)f_1[0.1, -0.46, -0.532]$

$= (0.2)[-0.532]$

$= -0.1064$

$$l_2 = h f_2\left[x_0+\frac{h}{2},\ y_0+\frac{k_1}{2},\ z_0+\frac{l_1}{2}\right]$$

$$=(0.2) f_2\left[0+\frac{0.2}{2},\ -0.4-\frac{0.12}{2},\ -0.6+\frac{0.136}{2}\right]$$

$$=(0.2) f_2\left[0.1,\ -0.46,\ -0.532\right]$$

$$=(0.2)\left[2(-0.532)-2(-0.46)\right]+e^{2(0.1)}\sin(0.1)$$

$$=(0.2)\left[-1.064+0.92+0.1294\right]$$

$$=-0.00292$$

$$k_3 = h f_1\left[x_0+\frac{h}{2},\ y_0+\frac{k_2}{2},\ z_0+\frac{l_2}{2}\right]$$

$$=(0.2) f_1\left[0+\frac{0.2}{2},\ -0.4-\frac{0.1064}{2},\ -0.6-\frac{0.00292}{2}\right]$$

$$=0.2 f_1\left[0.1,\ -04532,\ -0.60146\right]$$

$$=0.2\left[-0.60146\right]$$

$$=-0.1203$$

$$l_3 = h f_2\left[x_0+\frac{h}{2},\ y_0+\frac{k_2}{2},\ z_0+\frac{l_2}{2}\right]$$

$$=(0.2) f_2\left[0+\frac{0.2}{2},\ -0.4-\frac{0.1064}{2},\ -0.6-\frac{0.00292}{2}\right]$$

$$=(0.2) f_2\left[0.1,\ -0.4532,\ -0.60146\right]$$

$$=(0.2)\left[2(-0.60146)-2(-0.4532)+e^{2(0.1)}\sin(0.1)\right]$$

$$=(0.2)\left[-1.20292+0.9064+0.12194\right]$$

$$=-0.0105$$

$$k_4 = h f_1\left[x_0+h,\ y_0+k_3,\ z_0+l_3\right]$$

$$=(0.2) f_1\left[0+0.2,\ -0.4-0.1203,\ -0.6-0.0105\right]$$

$$=(0.2) f_1\left[0.2,\ -0.5203,\ -0.6105\right]$$

$$=(0.2)\left[-0.6105\right]$$

$$=-0.1221$$

$$l_4 = h f_2\left[x_0+h,\ y_0+k_3,\ z_0+l_3\right]$$

$$=(0.2) f_2\left[0+0.2,\ -0.4-0.1203,\ -0.6-0.0105\right]$$

$$=(0.2)\left[2(-0.6105)-2(-0.5203)+e^{2(0.2)}\sin(0.2)\right]$$

$$=(0.2)\left[-1.221+1.0406+0.29638\right]$$

$$=0.0825$$

$$\Delta y = \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$=\frac{1}{6}\left[-0.12+2(-0.1064)+2(-0.1203)+(-0.1221)\right]$$

$$=\frac{-1}{6}\left[0.12+2(0.1064)+2(0.1203)+0.1221\right]$$

$$=-0.1159$$

$$y_1 = y_0+\Delta y = -0.4-0.1159$$

$$=-0.5159$$

$$y(0.2) = -0.5159$$

$$\Delta z = \frac{1}{6}\left[l_1+2l_2+2l_3+l_4\right]$$

$$=\frac{1}{6}\left[0.136+2(-0.00292)+2(-0.0105)+0.0825\right]$$

$$=\frac{1}{6}\left[0.136-2(0.00292)-2(0.0105)+0.0825\right]$$

$$=0.03194$$

$$z_1 = z_0+\Delta z = -0.6+0.3194 = -0.2806$$

MILNE'S PREDICTOR AND CORRECTOR METHODS

milne's predictor corrector formulae

① predictor

$$y_{n+1},\ P = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

② Corrector

$$y_{n+1},\ C = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

① using milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

Soln:

$$y' = \frac{2-y^2}{5x}, \quad x_0 = 4 \quad x_1 = 4.1, \quad x_2 = 4.2, \quad x_3 = 4.3$$

$$x_4 = 4.4 \qquad y_0 = 1 \quad y_1 = 1.0049, \quad y_2 = 1.0097, \quad y_3 = 1.0143$$

$$y_1' = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0493$$

$$y_2' = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.0467$$

$$y_3' = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0452$$

By milne's predictor formula

$$y_4,\ P = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0493) - 0.0467 + 2(0.0452)\right]$$

$$= 1.01897$$

$$y_4' = \frac{2-y_4^2}{5(x_4)} = \frac{2-(1.01897)^2}{5(4.4)} = 0.0437$$

using

$$y_4,\ C = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.0437\right]$$

$$y_4, C = 1.01874$$

② Solve $y' = x - y^2$, $0 \le x \le 1$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ by milne's method to find $y(0.8)$ and $y(1)$.

Soln:-

Here $x_0 = 0$ $\quad y_0 = 0$

$x_1 = 0.2$ $\quad y_1 = 0.02$ $\qquad h = 0.2$

$x_2 = 0.4$ $\quad y_2 = 0.0795$

$x_3 = 0.6$ $\quad y_3 = 0.1762$

$x_4 = 0.8$ $\quad y_4 = ?$

$x_5 = 1$ $\quad y_5 = ?$

$$y' = f(x, y) = x - y^2 \quad \text{——} ①$$

By milne's predictor formula

$$y_{n+1, P} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$y_{4, P} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad \text{——} ②$$

from (1) $y' = x - y^2$

$y_1' = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$

$y_2' = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$

$y_3' = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5690$

$$y_{4, P} = 0 + \frac{4(0.2)}{3}\left[2(0.1996) - (0.3937) + 2(0.5690)\right]$$

$$= \frac{0.8}{3}\left[1.1435\right] = 0.3049$$

$y_4' = x_4 - y_4^2$

$= 0.8 - (0.3049)^2$

$= 0.707$

$$y_{4, C} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5690) + 0.707\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[3.3767\right]$$

$= 0.3046$

Corrected value of y at $x = 0.8$ is 0.3046

To find $y(1)$

$$y_{5,P} = y_1 + \frac{4h}{3}[2y_2' - y_3' + 2y_4']$$

$$= 0.02 + \frac{4(0.2)}{3}[2(0.3937) - (0.5690) + 2(0.707)]$$

$$= 0.02 + \frac{4(0.2)}{3}[1.6324]$$

$$= 0.02 + \frac{1.30592}{3} = 0.4553$$

$$y_{5,P}' = x_5 - y_5^2 = 1 - (0.4533)^2 = 0.7327$$

$$y_{5,C} = y_3 + \frac{h}{3}[y_3' + 4y_4' + y_5']$$

$$= (0.1762) + \frac{0.2}{3}[0.569 + 4(0.707) + 0.7327]$$

$$= 0.1762 + \frac{0.2}{3}[4.1297]$$

$$= 0.4515$$

Corrected value of y at $x = 1$ is 0.4515.

(3) Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$

The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R-K method of fourth order. Find $y(0.8)$ by milne's predictor-corrector method taking $h = 0.2$

Soln:

Here $x_0 = 0$ $\quad y_0 = 2$

$x_1 = 0.2$ $\quad y_1 = 2.073$

$x_2 = 0.4$ $\quad y_2 = 2.452$

$x_3 = 0.6$ $\quad y_3 = 3.023$

$$y' = f(x, y) = x^3 + y \quad \text{—— (1)}$$

By milne's predictor formula.

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y_{n-2}' - y_{n-1}' + 2y_n']$$

$$y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3] \quad \text{—— (2)}$$

From (1) $y' = x^3 + y$

$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$

$y_2' = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$

$y_3' = x_3^3 + y_3 = (0.6)^3 + 3.023 = 3.239$

$(2) \Rightarrow y_4, P = 2 + \frac{4(0.2)}{3}[2(2.081) - 2.516 + 2(3.239)]$

$= 2 + \frac{0.8}{3}[8.124]$

$= 2 + 2.1664$

$= 4.1664$

using milne's corrector formula.

$y_{n+1}, C = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$

$y_4, C = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$ —— (3)

$y_1' = x_1^3 + y_1$

$y_2' = x_2^3 + y_2 = 2.516$

$y_3' = x_3^3 + y_3 = 3.239$

$y_4' = x_4^3 + y_4 = (0.8)^3 + 4.1664 = 4.6784$

$(3) \Rightarrow y_4, C = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.6784]$

$= 2.452 + \frac{0.2}{3}[20.1504]$

$= 3.79536$

Corrected value of y at $x = 0.8$ is 3.79536.

ADAM'S PREDICTOR AND CORRECTOR METHODS

Predictor:

$$y_{n+1},\ P = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

Corrector:

$$y_{n+1},\ C = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

① Given $\frac{dy}{dx} = x^2(1+y)$ $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams - Bashforth method.

Soln:

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$

$y_0 = 1$, $y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$, $y_4 = ?$

By Adam's method

predictor:

$$y_{n+1},\ P = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

$$y_4,\ P = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \quad (1)$$

Here $y_0' = x_0^2(1+y_0) = (1)^2(1+1) = 2$

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$

$y_2' = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.60912$

$y_3' = x_3^2(1+y_3) = (1.3)^2[1+1.979] = 5.0345$

$(1) \Rightarrow y_4,\ P = 1.979 + \frac{0.1}{24}[55(5.0345) - 59(3.60912) + 37(2.70193) - 9(2)]$

$= 1.979 + \frac{0.1}{24}[276.8975 - 212.93808 + 99.91741 - 18]$

$= 1.979 + \frac{0.1}{24}[145.93083]$

$= 1.979 + 0.6080451$

$= 2.5870451$

$= 2.5871$ [correct to four decimal places] corrector method:

$y_{n+1}, c = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$

$y_4, c = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$

$y_4' = x_4^2(1+y_4) = (1.4)^2[1 + 2.5871] = 7.030716$

(2) $\Rightarrow y_4, c = 1.979 + \frac{0.1}{24}\left[9(7.030716) + 19(5.0345) - 5(3.60912) + 2.70193\right]$

$= 1.979 + \frac{0.1}{24}\left[63.276444 + 95.6555 - 18.0456 + 2.70193\right]$

$= 1.979 + \frac{0.1}{24}[143.58827]$

$= 1.979 + 0.592844$

$= 2.5772844$

$= 2.5773$ [correct to four decimal places]

②. using Adam's Bashforth method find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

Soln:

Given $y' = \frac{2-y^2}{5x}$. Let $h = 0.1$

Given $x_0 = 4$, $y_0 = 1$, $x_1 = 4.1$, $y_1 = 1.0049$,

$x_2 = 4.2$ $y_2 = 1.0097$, $x_3 = 4.3$ $y_3 = 1.0143$

Adams predictor formula is

$y_{n+1}, P = y_n + \frac{h}{24}\left[55y_n' - 59y'_{n-1} - 37y'_{n-2} - 9y'_{n-3}\right]$

Putting n=3 we have

$y_4, P = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

$y_0' = (y')(x_0, y_0) = \frac{2-y_0^2}{5x_0} = 0.05$

$y_1' = (y')(x_1, y_1) = \frac{2-y_1^2}{5x_1} = 0.0483$

$y_2' = (y')(x_2, y_2) = \frac{2-y_2^2}{5x_2} = 0.0467$

$y_3' = (y')(x_3, y_3) = \frac{2-y_3^2}{5x_3} = 0.0452$

using these values in (1) we get

$y_4, P = 1.0143 + \frac{0.1}{24}\left[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)\right]$

$$= 1.01413 + \frac{0.1}{24}[4.2731 - 3.2053] = 1.0186$$

$$y(4.4) = 1.0186$$

Adam's corrector formula is

$$y_{n+1,\,C} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

putting n=3 we get

Now $y'_4 = y'(x_4, y_4) = \dfrac{2-y_4^2}{5x_4} = 0.0437$

∴ (2) Becomes

$$y_{4,\,C} = 1.0143 + \frac{0.1}{24}[9(0.0437) + 19(0.0452) - 5(0.0467) + 0.0483]$$

$$= 1.0143 + \frac{0.1}{24} \times 1.0669 = 1.0187$$

$$\therefore y(4.4) = 1.0187$$ Ans

UNIT-V

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

1. Solve by finite difference method, the boundary value problem $y''(x) - y(x) = 2$ where $y(0) = 0$ and $y(1) = 1$, taking $h = 1/4$

Soln

Given: $y''(x) - y(x) = 2$ —— (1)

using the central difference approximation, we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\therefore (1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 2$$

$$y_{i-1} - [2 + h^2]\, y_i + y_{i+1} = 2h^2$$

$$y_{i-1} - [2 + 1/16]\, y_i + y_{i+1} = 2\left[\frac{1}{16}\right]$$

$$y_{i-1} - \frac{33}{16} y_i + y_{i+1} = 1/8$$

$$y_{i-1} - 2.0625 y_i + y_{i+1} = 0.125 \quad \text{—— (2)}$$

The boundary conditions are $y_0 = y(0) = 0$, $y_4 = y(1) = 1$

(i) To find $y_1 = y(1/4)$, $y_2 = y(2/4)$, $y_3 = y(3/4)$

$i = 1$ in (2) $\Rightarrow y_0 - 2.0625 y_1 + y_2 = 0.125$

$$-2.0625 y_1 + y_2 = 0.125 \quad [\because y_0 = 0] \quad \text{—— (3)}$$

$i = 2$ in (2) $\Rightarrow y_1 - 2.0625 y_2 + y_3 = 0.125$ —— (4)

$i = 3$ in (2) $\Rightarrow y_2 - 2.0625 y_3 + y_4 = 0.125$

$$y_2 - 2.0625 y_3 + 1 = 0.125$$

$$y_2 - 2.0625 y_3 = -0.875 \quad \text{—— (5)}$$

(3) $\times 1 \Rightarrow -2.0625 y_1 + y_2 = 0.125$

(4) $\times (2.0625) \Rightarrow 2.0625 y_1 - 4.2539 y_2 + 2.0625 y_3 = 0.2578$

(+) $-3.2539 y_2 + 2.0625 y_3 = 0.3828$ —— (6)

(5) $\times$ 3.2539 $\Rightarrow$ $3.2539y_2 - 6.7112y_3 = -2.8472$

(6) $\times$ 1 $\Rightarrow$ $-3.2539y_2 + 2.0625y_3 = 0.3828$

(+) $-4.6487y_3 = -2.4644$

$$y_3 = \frac{2.4644}{4.6487} = 0.5301$$

(5) $\Rightarrow$ $y_2 = 2.0625y_3 - 0.875$

$= (2.0625)(0.5301) - 0.875 = 0.2183$

(4) $\Rightarrow$ $y_1 = 2.0625y_2 - y_3 + 0.125$

$= (2.0625)(0.2183) - (0.5301) + 0.125 = 0.0451$

Hence the result is

$y_0 = y(0) = 0$ [Given]

$y_1 = y(1/4) = 0.0451$

$y_2 = y(2/4) = 0.2183$

$y_3 = y(3/4) = 0.5301$

$y_4 = y(4/4) = y(1) = 1$ [Given]

(2) Solve the equation

$y''(x) - xy(x) = 0$ for $y(x_i)$, $x_i = 0, 1/3, 2/3$, given

$y(0) + y'(0) = 1$ and $y(1) = 1$

Soln:

The finite difference equation is

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = x_i y_i$$

(i.e) $y_{i-1} - (2 + \frac{1}{9}x_i)y_i + y_{i+1} = 0$ —— (1)

Since $h = 1/3$

putting $i = 0, 1, 2$ in (1) we have

$y_{-1} - 2y_0 + y_1 = 0$ —— (2)

$y_0 - \frac{55}{27}y_1 + y_2 = 0$ —— (3)

and $y_1 - \frac{56}{27} y_2 + y_3 = 0$ —— (4)

The first boundary condition is

$$y_0 + \frac{y_1 - y_{-1}}{2h} = 1 \qquad \left(\because y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \right)$$

$$2y_0 + 3(y_1 - y_{-1}) = 2$$

$$y_{-1} = \frac{2y_0 + 3y_1 - 2}{3} \quad \text{—— (5)}$$

The second boundary condition is

$$y_3 = 1$$

using (5) and (6) in equations (2), (3) and (4) we have

$$-2y_0 + 3y_1 = 1 \quad \text{—— (7)}$$

$$y_0 - \frac{57}{27} y_1 + y_2 = 0 \quad \text{—— (8)}$$

and $y_1 - \frac{56}{27} y_2 + 1 = 0$ —— (9)

solving equations (7), (8) and (9) we get

$$y_0 = y(0) = -\frac{82}{83} = -0.9880$$

$$y_1 = y(1/3) = \frac{-27}{83} = -0.3253$$

$$y_2 = y(2/3) = \frac{27}{83} = 0.3253$$

Finite Difference Solution of one dimensional Heat equation By implicit and Explicit methods

classification of partial differential equation of Second order

	Elliptic Type	parabolic Type	Hyperbolic Type
1.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace Eqn in two dimension)	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$ (one dimensional heat equation)	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2}$ (one dimensional wave equation)
2.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (poisson's equation)		

Example:- Classify the following partial equations:-

(i) $u_{xx} + 4u_{xy} + (x^2+4y^2)u_{yy} = \sin(x+y)$

(ii) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

Soln:

(i) Here, $A = 1$, $B = 4$, $C = (x^2+4y^2)$

$$B^2 - 4AC = 16 - 4(x^2+4y^2)$$
$$= 4[4-x^2-4y^2]$$

The equation is elliptic if $4 - x^2 - 4y^2 < 0$

$$x^2 + 4y^2 > 4$$
$$\frac{x^2}{4} + \frac{y^2}{1} > 1$$

$\therefore$ It is elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

It is hyperbolic inside, the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

(ii) Here, $A = x+1$, $B = -2(x+2)$, $C = x+3$

$$B^2 - 4AC = 4(x+2)^2 - 4(x+1)(x+3)$$
$$= 4[1] = 4 > 0$$

$\therefore$ The equation is hyperbolic at all points of the region.

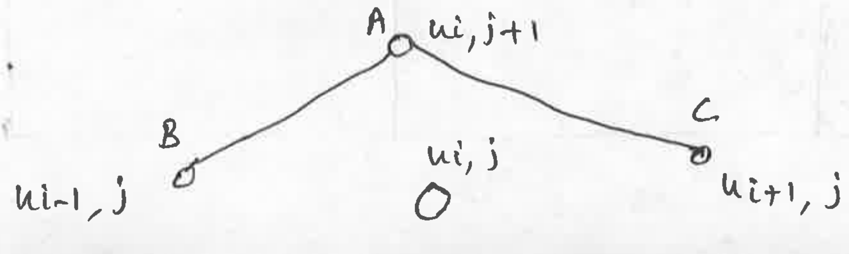
BENDER - SCHMIDT'S DIFFERENCE EQUATION

Corresponding to the parabolic equation $\left(\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\right)$ (one dimensional heat equation) [Explicit method]

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1\,j} + u_{i+1\,j}\right]$$

when $\lambda = \frac{1}{2} = \frac{k}{ah^2}$ (ie) $k = \frac{a}{2}h^2$

This is valid only if $k = \frac{a}{2}h^2$



① Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t)=0$, $u(4,t)=0$, $u(x,0)=x(4-x)$.
Assume $h=1$. Find the values of u upto $t=5$.

Soln:
$u_{xx} = a u_t$ $\therefore a=2$

To use Bender-Schmidt's equation, $k = \frac{a}{2}h^2 = 1$

Step-size in time $= k = 1$. The values of $u_{i,j}$ are tabulated below.

x - direction →

j \ i (t-direction ↓)	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

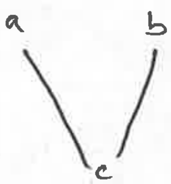
← $u(x,0) = x(4-x)$

The range for x is $(0,4)$; for $t: (0,5)$

$u(x,0) = x(4-x)$ this gives $u(0,0)=0$, $u(1,0)=3$, $u(2,0)=4$, $u(3,0)=3$, $u(4,0)=0$.

For all t, $x=0$, $u=0$ and for all t at $x=4$, $u=0$

using these values we fill up column under $x=0$, $x=4$ and row against $t=0$



This means $c = \frac{a+b}{2}$

The values of u at $t=1$ are written by seeing the values of u at $t=0$ and using the average formula.

②. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$, $f(0,t) = f(5,t) = 0$, $f(x,0) = x^2(25-x^2)$, find f in the range taking $h=1$ and upto 5 seconds.

Soln:
By using method, $k = \frac{a}{2}h^2$

Here $a=1$, $h=1$, $\therefore k = \frac{1}{2}$

Step-size of time $= \frac{1}{2}$

Step-size of $x = 1$.

$f(0,0) = 0, \quad f(1,0) = 24, \quad f(2,0) = 84, \quad f(3,0) = 144, \quad f(4,0) = 144, \quad f(5,0) = 0$

We have

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

→ x direction

↓ t-direction

$j \backslash i$	0	1	2	3	4	5
0	0	24	84	144	144	0
$\frac{1}{2}$	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0
4.5	0	11.4843	17.0625	18.5859	10.5469	0
5	0	8.5312	15.0351	13.8047	9.2929	0

CRANK-NICOLSON'S DIFFERENCE EQUATION, CORRESPONDING TO THE PARABOLIC EQUATION.

Formula

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

① Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0 < x < 5$ $t \geq 0$ given that $u(x,0) = 20$, $u(0,t) = 0$ $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson method.

Soln:

Here $a = 1$ $h = 1$ $\lambda = \frac{k}{ah^2}$

A convenient choice of λ makes the Crank-Nicholson difference scheme simple. Setting $\lambda = 1$, (ie) $k = ah^2$, $k = 1$

The Crank - Nicholson formula reduces to

$$u_{i,\,j+1} = \frac{1}{4}\left[u_{i-1,\,j+1} + u_{i+1,\,j+1} + u_{i-1,\,j} + u_{i+1,\,j}\right]$$

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

$u_1 = \frac{1}{4}\left[0+20+0+u_2\right]$

$= \frac{1}{4}\left[20+u_2\right]$

$4u_1 = 20+u_2$

$4u_1 - u_2 = 20$ —— ①

$u_2 = \frac{1}{4}\left[20+20+u_1+u_3\right]$

$= \frac{1}{4}\left[40+u_1+u_3\right]$

$4u_2 = 40+u_i+u_3$

$u_1+u_3-4u_2 = 40$ —— ②

$u_1 - 4u_2 + u_3 = -40$ —— ②

$u_3 = \frac{1}{4}\left[20+20+u_2+u_4\right]$

$4u_3 = 40+u_2+u_4$

$u_2 - 4u_3 + u_4 = -40$ —— ③

$u_4 = \frac{1}{4}\left[20+100+u_3+100\right]$

$4u_4 = 220+u_3$

$u_3 - 4u_4 = -220$ —— ④

① × 1 ⇒ $4u_1 - u_2 = 20$

② × 4 ⇒ $4u_1 - 16u_2 + 4u_3 = -160$

(−) $15u_2 - 4u_3 = 180$ —— ⑤

③ × 4 ⇒ $4u_2 - 16u_3 + 4u_4 = -160$

④ × 1 ⇒ $u_3 - 4u_4 = -220$

(+) $4u_2 - 15u_3 = -380$ —— ⑥

(5) $\times 4 \Rightarrow \quad 60u_2 - 16u_3 = 720.$

(8) $\times 15 \Rightarrow \quad 60u_2 - 225u_3 = -5700$

(-) $\quad 209u_3 = 6420$

$$u_3 = 30.72$$

(4) $\Rightarrow \quad 4u_4 = 220 + u_3 = 220 + 30.72$

$$u_4 = 62.68$$

(3) $\Rightarrow \quad u_2 = 4u_3 - u_4 - 40$

$= 4(30.720) - 62.68 - 40$

$= 20.2$

(1) $\Rightarrow \quad u_1 = \frac{1}{4}[20 + u_2]$

$= \frac{1}{4}[20 + 20.2]$

$= 10.05$

$\therefore$ The values are 10.05, 20.2, 30.72, 62.68

(2) Solve by Crank-Nicholson method the equation $u_{xx} = u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$, for two time steps.

Soln:-

x ranges from 0 to 1 take $h = \frac{1}{4}$; here $a = 1$.

$\therefore k = ah^2$ to use simple form

$$k = 1\left(\tfrac{1}{4}\right)^2 = \frac{1}{16}$$

we use $u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$ —(1)

$x \rightarrow$ direction

$t \downarrow$

	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$
$\frac{3}{16}$	0				$\frac{3}{16}$

Let the unknowns be represented by $u_1, u_2, u_3 \ldots$

The boundary conditions are marked in the table against $t=0$, $x=0$, and $x=1$

using the scheme (1),

$u_1 = \frac{1}{4}(0+0+0+u_2)$ $\qquad u_1 = \frac{1}{4}u_2$

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $\qquad u_2 = \frac{1}{4}(u_1+u_3)$

$u_3 = \frac{1}{4}(0+0+u_2+\frac{1}{16})$ $\qquad u_3 = \frac{1}{4}(u_2+\frac{1}{16})$

Solving the three equations given by (2), (3) (4) we get u_1, u_2, u_3

Substitute u_3, u_1 values in (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}\left(u_2+\frac{1}{16}\right)\right]$$

$u_2 = \frac{1}{224}$ 0.0045, $u_1 = \frac{1}{896} = 0.0011$, $u_3 = 0.0168$

Similarly u_4, u_5, u_6 can be got again getting 3 equations in 3 unknowns u_4, u_5, u_6.

we get $u_4 = 0.005899$ $u_5 = 0.01913$, $u_6 = 0.05277$

In solving the three equations (2), (3), (4) we could have used Gauss-Seidel method also. The iterated values are noted below.

u_1	-	0.125	0.0391	0.0059	0.0017	0.0012	0.0011	0.0011
u_2	0.5	0.1563	0.0235	0.0069	0.0048	0.0045	0.0045	0.0045
u_3	0.5	0.0547	0.0215	0.0174	0.0168	0.0168	0.0168	0.0168

ONE DIMENSIONAL WAVE EQUATION

Formula :- $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

① Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0,t)=0$ $y(1,t)=0$, $y_t(x,0)=0$ and $y(x,0) = 10 + x(1-x)$.

Soln:

Here $a=1$, $h=0.1$, $k = \frac{h}{a} = 0.1$

Formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

From $y(0,t) = 0 \Rightarrow$ y along $x=0$ are all zero

From $y(1,t) = 0 \Rightarrow$ y along $x=1$ are all zero.

$y(x,0) = 10 + x(1-x)$

$y(0.1,0) = 10 + (0.1)(1-0.1) = 10.09$

$y(0.2,0) = 10 + (0.2)(1-0.2) = 10.16$

$y(0.3,0) = 10 + (0.3)(1-0.3) = 10.21$

$y(0.4,0) = 10 + (0.4)(1-0.4) = 10.24$

$y(0.5,0) = 10 + (0.5)(1-0.5) = 10.25$

$y(0.6,0) = 10 + (0.6)(1-0.6) = 10.24$

$y(0.7,0) = 10 + (0.7)(1-0.7) = 10.21$

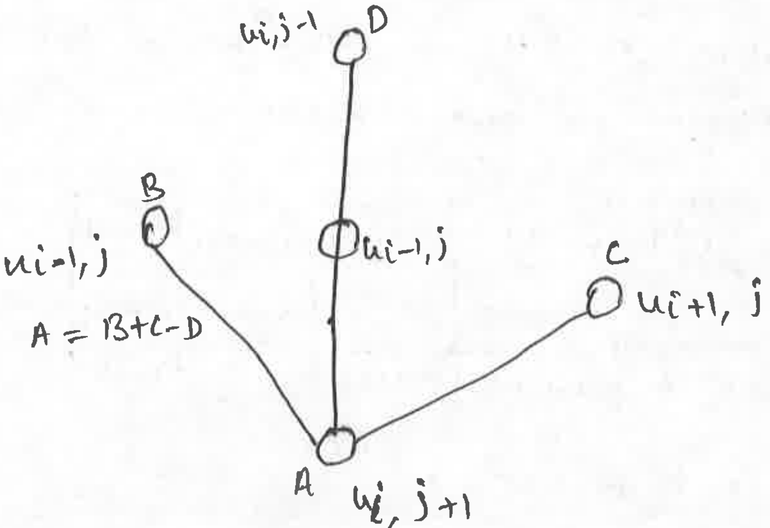
$y(0.8,0) = 10 + (0.8)(1-0.8) = 10.16$

$y(0.9,0) = 10 + (0.9)(1-0.9) = 10.09$

Now, we fill up the row $t=0$ using the above values

$$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$$

Now we draw the table for what we require



$\rightarrow x$

$t \backslash x$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	10.0
0.1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	10.08	0.0
0.2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0.0
0.3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0.0
0.4	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0.02	0.0
0.5	0	0	0	0	0	0	0	0	0	0	0

$\downarrow$ t

5. Approximate the solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = \sin 2\pi x$, $0 \le x \le 1$ and $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \le x \le 1$ with $\Delta x = 0.25$ and $\Delta t = 0.25$ for 3 time steps.

Soln:

Here $a = 1$, $h = 0.25$, $k = 0.25$

0.25 \ 0.25	0	0.25	0.5	0.75	1
0	0	1	0	-1	0
0.25	0	0	0	0	0
0.5	0	-1	0	1	0
0.75	0	0	0	0	0
1	0	1	0	-1	0

$u(x,0) = \sin 2\pi x$

$u(0.25, 0) = \sin \pi/2 = 1$

$u(0.5, 0) = \sin \pi = 0$

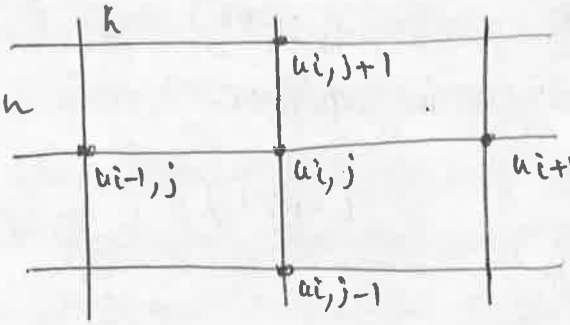
$u(0.75, 0) = \sin \frac{3\pi}{2} = -1$

TWO DIMENSIONAL LAPLACE EQUATION

ELLIPTIC EQUATIONS

$$\therefore u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

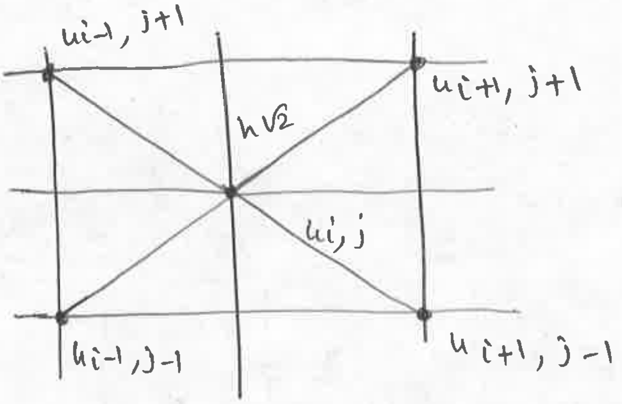
This is called standard five point formula



Central value = Average of the other four values.

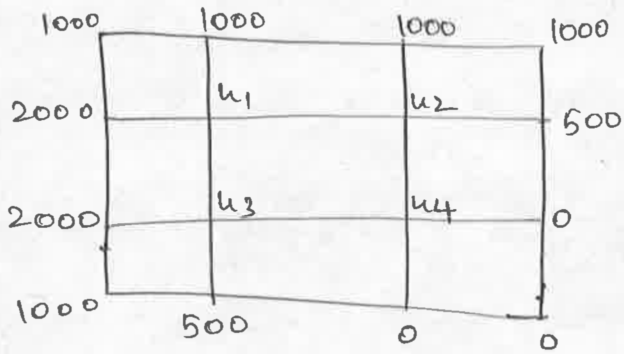
Diagonal five-point formula.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



① Obtain a finite difference scheme to solve the Laplace equation

Solve $\nabla^2 u = 0$ at the pivotal points in the square shown fitted with square mesh. Use Leibmann's iteration procedure. (5 iteration only)



Solution:-

we can assume some value for u_4 (or any other u) and proceed iterative procedure; we can take $u_4 = 0$ and proceed

(or) we take a value of $u_4 = 400$ (Guess this seeing the values of u on the vertical line through u_2, u_4).

Rough values:-

$u_1 = (1000 + 2000 + 1000 + 400)/4 = 1100$ (DFPF)

$u_2 = \frac{1}{4}(u_1 + u_4 + 1500) = 750$ (SFPF)

$u_3 = \frac{1}{4}(u_1 + u_4 + 2500) = 1000$ (SFPF)

$u_4 = \frac{1}{4}(u_2 + u_3) = 437.5$ (SFPF)

First iteration: Here after we apply only SFPF

$u_1^{(1)} = \frac{1}{4}(750 + 1000 + 3000) = 1187.5$

$u_2^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 1500) = 781.25$

$u_3^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 2500) = 1031.25$

$u_4^{(1)} = \frac{1}{4}(781.25 + 1031.25) = 453.125$

Second iteration:-

$u_1^{(2)} = \frac{1}{4}(781.25 + 1031.25 + 3000) = 1203.125$

$u_2^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 1500) = 789.1$

$u_3^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 2500) = 1039.1$

$u_4^{(2)} = \frac{1}{4}(789.1 + 1039.1) = 457.1$

Third iteration:-

$u_1^{(3)} = \frac{1}{4}(789.1 + 1039.1 + 3000) = 1207.1$

$u_2^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 1500) = 791.1$

$u_3^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 2500) = 1041.1$

$u_4^{(3)} = \frac{1}{4}(791.1 + 1041.1) = 458.1$

Fourth iteration:-

$u_1^{(4)} = \frac{1}{4}(791.1 + 1041.1 + 3000) = 1208.1$

$u_2^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 1500) = 791.6$

$u_3^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 2500) = 1041.6$

$u_4^{(4)} = \frac{1}{4}(791.6 + 1041.6) = 458.3$

Fifth iteration:-

$u_1^{(5)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3$

$u_2^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7$

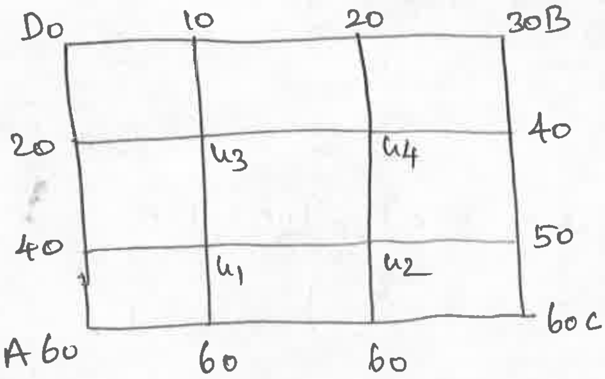
$u_3^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7$

$u_4^{(5)} = \frac{1}{4}(791.7 + 1041.7) = 458.7$

we are getting result correct to one decimal place. Further the increases in the value is < 0.1

$\therefore u_1 = 1208.1,\ u_2 = 791.7,\ u_3 = 1041.7,\ u_4 = 458.4$

⑤ Solve:- $\Delta^2 u = 0$, the boundary conditions are given below. (give only three iteration).



Rough values: Let $u_4 = 0$

$u_1 = \frac{1}{4}[60 + u_4 + 20 + 60] = 35$ (DFPF)

$u_2 = \frac{1}{4}[u_1 + u_4 + 50 + 60] = 36.25$ (SFPF)

$u_3 = \frac{1}{4}[10 + 20 + u_1 + u_4] = 16.25$ (SFPF)

$u_4 = \frac{1}{4}[20 + u_3 + u_2 + 40] = 28.125$ (SFPF)

First iteration [Here after we apply only SFPF]

$u_1^{(1)} = \frac{1}{4}[40 + 60 + 16.25 + 36.25] = 38.125$

$u_2^{(1)} = \frac{1}{4}[60 + 50 + 38.125 + 28.125] = 44.0625$

$u_3^{(1)} = \frac{1}{4}[20 + 10 + 38.125 + 28.125] = 24.0625$

$u_4^{(1)} = \frac{1}{4}[20 + 40 + 24.0625 + 44.0625] = 32.0313$

Second iteration:-

$u_1^{(2)} = \frac{1}{4}[60 + 40 + 24.0625 + 44.0625] = 42.0313$

$u_2^{(2)} = \frac{1}{4}[60 + 50 + 42.0313 + 32.0313] = 46.0157$

$u_3^{(2)} = \frac{1}{4}[20 + 10 + 42.0313 + 32.0313] = 26.0157$

$u_4^{(2)} = \frac{1}{4}[20 + 40 + 26.0157 + 46.0157] = 33.0079$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}[60+40+46.0157+26.0157] = 43.0079$$

$$u_2^{(3)} = \frac{1}{4}[60+50+43.0079+33.0079] = 46.5040$$

$$u_3^{(3)} = \frac{1}{4}[20+10+43.0079+33.0079] = 26.5040$$

$$u_4^{(3)} = \frac{1}{4}[20+40+46.5040+26.5040] = 33.252$$

(3) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $|x| < 1$, $|y| < 1$ with $h = \frac{1}{2}$ and

$u(x \pm 1) = x^2$, $u(\pm 1, y) = y^2$

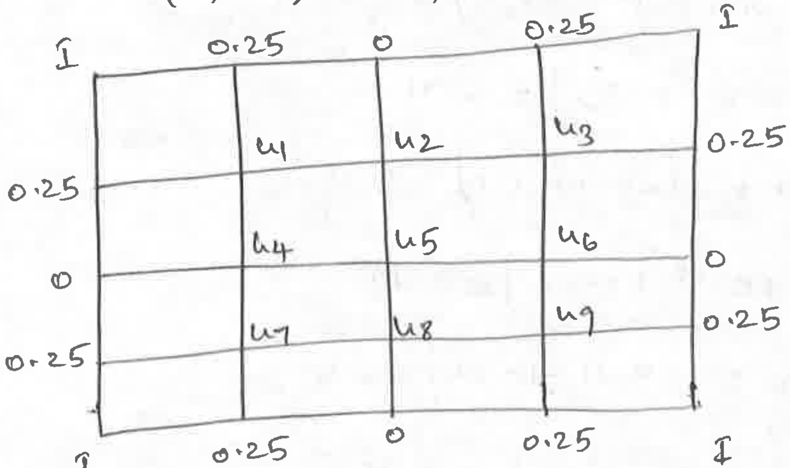
(i) $u(x, 1) = x^2$ $\quad -1 < x < 1$

(ii) $u(x, -1) = x^2$, $\quad -1 < x < 1$

(iii) $u(1, y) = y^2$, $\quad -1 < y < 1$

(iv) $u(-1, y) = y^2$, $\quad -1 < y < 1$

Soln:- Given $u(x, \pm 1) = x^2$, $u(\pm 1, y) = y^2$



Let the values of u at the 9 interior grid points be $u_1, u_2, u_3 \dots u_9$

We use Liebman's iteration method to find the value of $u_1, u_2, u_3 \dots u_9$

Step 1:- To find the 9 rough values of $u_1, u_2, u_3 \dots u_9$.

$$u_5 = \frac{1}{4}[0+0+0+0] = 0 \; \{SFPF\}$$

$$u_1 = \frac{1}{4}[0+0+0+1] = \frac{1}{4} \; [DFPF]$$

$$u_3 = \frac{1}{4}[1+0+0+0] = \frac{1}{4} \; [DFPF]$$

$u_7 = \frac{1}{4}[1+0+0+0] = \frac{1}{4}$ [DFPF]

$u_9 = \frac{1}{4}[1+0+0+0] = \frac{1}{4}$ [DFPF]

Now we find the other four values using [SFPF]

$$u_2 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_4 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_6 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

$$u_8 = \frac{1}{4}[0+0+0.25+0.25] = 0.125$$

Step: 2 First iteration :- In all further calculations we use SFPF and the latest available values

$$u_1^{(1)} = \frac{1}{4}[0.25+0.25+0.125+0.125] = 0.19$$

$$u_2^{(1)} = \frac{1}{4}[0+0.19+0.25+0] = 0.11$$

$$u_3^{(1)} = \frac{1}{4}[0.25+0.25+0.11+0.125] = 0.18$$

$$u_4^{(1)} = \frac{1}{4}[0+0.19+0+0.25] = 0.11$$

$$u_5^{(1)} = \frac{1}{4}[0.11+0.11+0.125+0.125] = 0.12$$

$$u_6^{(1)} = \frac{1}{4}[0+0.12+0.18+0.25] = 0.14$$

$$u_7^{(1)} = \frac{1}{4}[0.25+0.25+0.11+0.25] = 0.18$$

$$u_8^{(1)} = \frac{1}{4}[0+0.12+0.18+0.25] = 0.14$$

$$u_9^{(1)} = \frac{1}{4}[0.25+0.25+0.14+0.14] = 0.20$$

Second iteration:-

$$u_1^{(2)} = \frac{1}{4}[0.25+0.25+0.11+0.11] = 0.18$$

$$u_2^{(2)} = \frac{1}{4}[0+0.18+0.\cancel{12}+0.18] = 0.12$$

$$u_3^{(2)} = \frac{1}{4}[0.25+0.25+0.12+0.14] = 0.19$$

$$u_4^{(2)} = \frac{1}{4}[0+0.18+0.12+0.18] = 0.12$$

$u_5^{(2)} = \frac{1}{4}[0.12+0.12+0.14+0.14] = 0.13$

$u_6^{(2)} = \frac{1}{4}[0.13+0+0.19+0.2] = 0.13$

$u_7^{(2)} = \frac{1}{4}[0.25+0.12+0.14+0.25] = 0.19$

$u_8^{(2)} = \frac{1}{4}[0+0.19+0.2+0.13] = 0.13$

$u_9^{(2)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$

Third iteration:-

$u_1^{(3)} = \frac{1}{4}[0.25+0.25+0.12+0.12] = 0.19$

$u_2^{(3)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$

$u_3^{(3)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$

$u_4^{(3)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$

$u_5^{(3)} = \frac{1}{4}[0.13+0.13+0.13+0.13] = 0.13$

$u_6^{(3)} = \frac{1}{4}[0.13+0+0.19+0.19] = 0.13$

$u_7^{(3)} = \frac{1}{4}[0.25+0.13+0.13+0.25] = 0.19$

$u_8^{(3)} = \frac{1}{4}[0+0.19+0.19+0.13] = 0.13$

$u_9^{(3)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19.$

Fourth iteration:-

$u_1^{(4)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$

$u_2^{(4)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$

$u_3^{(4)} = \frac{1}{4}[0.25+0.25+0.13+0.13] = 0.19$

$u_4^{(4)} = \frac{1}{4}[0+0.19+0.13+0.19] = 0.13$

$u_5^{(4)} = \frac{1}{4}[0.13+0.13+0.13+0.13] = 0.13$

$u_6^{(4)} = \frac{1}{4}[0.13+0+0.19+0.19] = 0.13$

$u_7^{(4)} = \frac{1}{4}[0.25+0.13+0.13+0.25] = 0.19$

$u_8^{(4)} = \frac{1}{4}[0 + 0.19 + 0.19 + 0.13] = 0.13$

$u_9^{(4)} = \frac{1}{4}[0.25 + 0.25 + 0.13 + 0.13] = 0.19$

Step;3:

Since in the third and fouth iterations all the values of $u_{i,j}$ at the grid points are same, the iteration process is stoped.

Hence

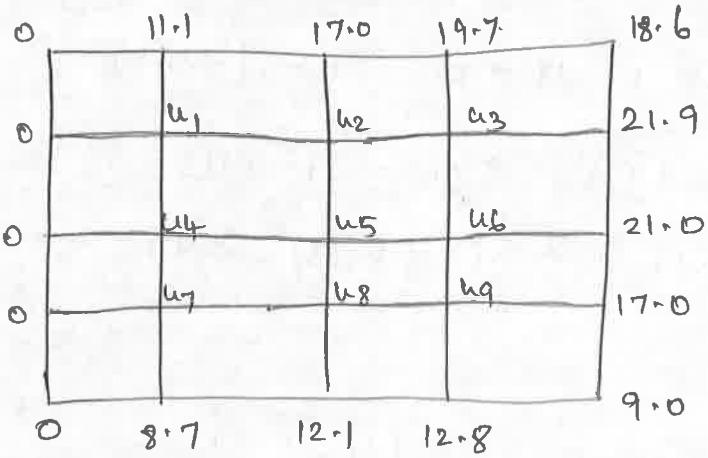
$u_1 = 0.19 \quad u_2 = 0.13, \quad u_3 = 0.19$

$u_4 = 0.13, \quad u_5 = 0.13 \quad u_6 = 0.13$

$u_7 = 0.19 \quad u_8 = 0.13 \quad u_9 = 0.19$

[correct to 2 places of decimals]

④ Find by the Liebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown in the following figure.



Solution:-

Since u is harmonic it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in the square} \quad —①$$

Let the interior values of u at the grid points be $u_1, u_2, u_3 \ldots u_9$. we will find the values of u at the interior mesh points as explained in the previous article. we will first find the rough values of u and then proceed to refine them.

Finding rough values

$$u_5 = \frac{1}{4}[0+17.0+21.0+12.1] = 12.5 \quad (SFPF)$$

$$u_1 = \frac{1}{4}[0+12.5+0+17.0] = 7.4 \quad (DFPF)$$

$$u_3 = \frac{1}{4}[12.5+18.6+17.0+1.0] = 17.3 \quad (DFPF)$$

$$u_7 = \frac{1}{4}[12.5+0+0+12.1] = 6.2 \quad (DFPF)$$

$$u_9 = \frac{1}{4}[12.5+9.0+12.1+21.0] = 13.7 \quad (DFPF)$$

$$u_2 = \frac{1}{4}[17.0+12.5+7.4+17.3] = 13.6 \quad (SFPF)$$

$$u_4 = \frac{1}{4}[7.4+6.2+0+12.5] = 6.5 \quad (SFPF)$$

$$u_6 = \frac{1}{4}[12.5+21.0+17.3+13.7] = 16.1 \quad (SFPF)$$

$$u_8 = \frac{1}{4}[12.5+12.1+6.2+13.7] = 11.1 \quad (SFPF)$$

Now, we have got the rough values at all interior grid points and already we possess the boundary values at the lattice points. We will now improve the values by using always SFPF

First iteration: (we obtain all values by SFPF)

$$u_1^{(1)} = \frac{1}{4}[0+11.1+u_2+u_4] = \frac{1}{4}[0+11.1+13.6+6.5] = 7.8$$

$$u_2^{(1)} = \frac{1}{4}[17.0+12.5+7.8+17.3] = 13.7$$

$$u_3^{(1)} = \frac{1}{4}[13.7+21.9+19.7+16.1] = 17.9$$

$$u_4^{(1)} = \frac{1}{4}[0+12.5+7.8+6.2] = 6.6$$

$$u_5^{(1)} = \frac{1}{4}[13.7+11.1+6.6+16.1] = 11.9$$

$$u_6^{(1)} = \frac{1}{4}[17.9+13.7+11.9+21.0] = 16.1$$

$$u_7^{(1)} = \frac{1}{4}[6.6+8.7+0+11.1] = 6.6$$

$$u_8^{(1)} = \frac{1}{4}[11.9+12.1+6.6+13.7] = 11$$

$$u_9^{(1)} = \frac{1}{4}[16.1+12.8+17.0+11.1] = 14.3$$

Second iteration:-

$$u_1^{(2)} = \frac{1}{4}[0 + 11.1 + 13.7 + 6.6] = 7.9$$

$$u_2^{(2)} = \frac{1}{4}[17.0 + 17.9 + 7.9 + 11.9] = 13.7$$

$$u_3^{(2)} = \frac{1}{4}[13.7 + 21.9 + 19.7 + 16.1] = 17.9$$

$$u_4^{(2)} = \frac{1}{4}[0 + 11.9 + 7.9 + 6.6] = 6.6$$

$$u_5^{(2)} = \frac{1}{4}[13.7 + 11.1 + 6.6 + 16.1] = 11.9$$

$$u_6^{(2)} = \frac{1}{4}[17.9 + 14.3 + 11.9 + 21.0] = 16.3$$

$$u_7^{(2)} = \frac{1}{4}[6.6 + 8.7 + 0 + 11.1] = 6.6$$

$$u_8^{(2)} = \frac{1}{4}[11.9 + 12.1 + 6.6 + 14.3] = 11.2$$

$$u_9^{(2)} = \frac{1}{4}[16.3 + 12.8 + 17.0 + 11.2] = 14.3$$

Third iteration:-

$$u_1^{(3)} = \frac{1}{4}[0 + 11.1 + 13.7 + 6.6] = 7.9$$

$$u_2^{(3)} = \frac{1}{4}[17.0 + 17.9 + 7.9 + 11.9] = 13.7$$

$$u_3^{(3)} = \frac{1}{4}[13.7 + 21.9 + 19.7 + 16.3] = 17.9$$

$$u_4^{(3)} = \frac{1}{4}[0 + 11.9 + 7.9 + 6.6] = 6.6$$

$$u_5^{(3)} = \frac{1}{4}[13.7 + 11.2 + 6.6 + 16.3] = 11.9$$

$$u_6^{(3)} = \frac{1}{4}[17.9 + 14.3 + 11.9 + 21.0] = 16.3$$

$$u_7^{(3)} = \frac{1}{4}[6.6 + 8.7 + 0 + 11.2] = 6.6$$

$$u_8^{(3)} = \frac{1}{4}[11.9 + 12.1 + 6.6 + 14.3] = 11.2$$

$$u_9^{(3)} = \frac{1}{4}[16.3 + 12.8 + 17.0 + 11.2] = 14.3$$

The third iteration values are same as the corresponding values of the second iteration. Hence we stop the procedure and accept.

$u_1 = 7.9$, $u_2 = 13.7$, $u_3 = 17.9$, $u_4 = 6.6$, $u_5 = 11.9$, $u_6 = 16.3$, $u_7 = 6.6$

$u_8 = 11.2$, $u_9 = 14.3$

TWO DIMENSIONAL POISSON EQUATION

Poisson equation

Any equation of the form $\nabla^2 u = f(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad -① $$

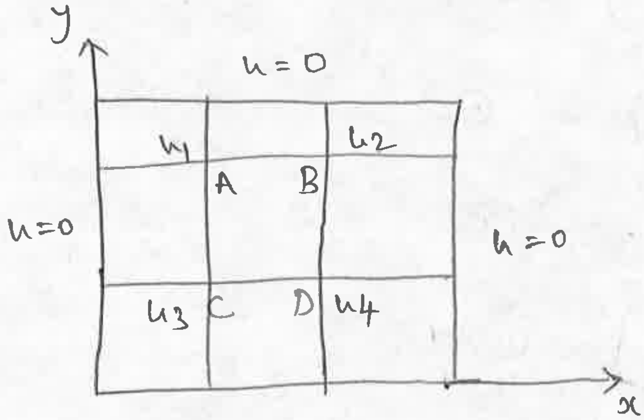
is called as poisson's equation where $f(x, y)$ is a function of x and y only

Formula

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad -②$$

① Solve the poisson equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0, y=0, x=3\ y=3$ with $u=0$ on the boundary, taking $h=1$.

Soln:



Let the values of u at the four mesh points A, B C and D be u_1, u_2, u_3, u_4 respectively. The differential equation is

$$\nabla^2 u = -10(x^2+y^2+10) \quad —①$$

Replacing $\nabla^2 u$ by the finite difference expressions and putting $x = ih$, $y = ih\ (h=1)$ in (1) we get

$$u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = -10(i^2+j^2+10)$$
$$—②$$

Applying the formula (1) at A [Where $i=1,\ j=2$]

$$0 + 0 + u_2 + u_3 - 4u_1 = -10(1+4+10)$$

$$u_2 + u_3 - 4u_1 = -150 \quad —③$$

Applying the formula (1) at B where $i=2, j=2$

$$u_1 + 0 + 0 + u_4 - 4u_2 = -10(4+4+10)$$

$$u_1 + u_4 - 4u_2 = -180 \quad \text{--- (4)}$$

Applying the formula (1) at C where $i=1, j=1$

$$0 + u_1 + u_4 + 0 - 4u_3 = -10(1+1+10)$$

$$u_1 + u_4 - 4u_3 = -120 \quad \text{--- (5)}$$

Applying the formula (1) at D where $i=2, j=1$

$$u_3 + u_2 + 0 + 0 - 4u_4 = -10(4+1+10)$$

$$u_3 + u_2 - 4u_4 = -150 \quad \text{--- (6)}$$

$$u_1 = \frac{1}{4}[u_2 + u_3 + 150] \quad \text{--- (7)}$$

$$u_2 = \frac{1}{4}[u_1 + u_4 + 180] \quad \text{--- (8)}$$

$$u_3 = \frac{1}{4}[u_1 + u_4 + 120] \quad \text{--- (9)}$$

$$u_4 = \frac{1}{4}[u_2 + u_3 + 150] \quad \text{--- (10)}$$

From (7) and (10)

we find that $u_4 = u_1$

So it is enough if we find u_1, u_2, u_3

we start the iteration by putting $u_2 = 0 = u_3$ in (7)

we get $u_1^{(1)} = \frac{150}{4} = 37.5$

putting $u_1 = 37.5 = u_4$ in (8) and (9) we get

$$u_2^{(1)} = \frac{1}{4}(75+180) = \frac{225}{4} = 63.75$$

$$u_3^{(1)} = \frac{1}{4}(75+120) = \frac{195}{4} = 48.7$$

For the second iteration, we have

$$u_1^{(2)} = \frac{1}{4}(63.75 + 48.75 + 150) = \frac{262.5}{4} = 65$$

$$u_2^{(2)} = \frac{1}{4}(65 + 65 + 180) = \frac{310}{4} = 80$$

$$u_3^{(2)} = \frac{1}{4}(65 + 65 + 120) = \frac{250}{4} = 60$$

For the third iteration, we have

$$u_1^{(3)} = \frac{1}{4}(80 + 60 + 150) = \frac{290}{4} = 70$$

$$u_2^{(3)} = \frac{1}{4}(70 + 70 + 180) = \frac{320}{4} = 80$$

$$u_3^{(3)} = \frac{1}{4}[70 + 70 + 120] = \frac{260}{4} = 65$$

For fourth iteration, we have

$$u_1^{(4)} = \frac{1}{4}(80 + 65 + 150) = \frac{295}{4} = 75$$

$$u_2^{(4)} = \frac{1}{4}(75 + 75 + 180) = \frac{330}{4} = 82.5$$

$$u_3^{(4)} = \frac{1}{4}(75 + 75 + 120) = \frac{270}{4} = 67.5$$

For the fifth iteration, we have

$$u_1^{(5)} = \frac{1}{4}(82 + 67.5 + 100) = \frac{300}{4} = 75$$

$$u_2^{(5)} = \frac{1}{4}(75 + 75 + 180) = \frac{330}{4} = 82.5$$

$$u_3^{(5)} = \frac{1}{4}(75 + 75 + 120) = \frac{270}{4} = 67.5$$

we find that these values are the same as in the fourth iteration.

$u_1 = 75$, $u_2 = 82.5$, $u_3 = 67.5$, $u_4 = 75$ [since $u_4 = u_1$]

5) solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u=0$ on the four boundaries dividing the square into 16 subsquares of length 1 unit.

Solution:

Here $h=1$. The region of solution of the given Laplace equation with the boundary values are given in the table.

0	0	0	0	0
0	u_1	u_2	u_3	0
0	u_4	u_5	u_6	0
0	u_7	u_8	u_9	0
0	0	0	0	0

Let $u_1, u_2, u_3 \ldots u_9$ be the values of u at the interior grid points.

Choose coordinate system with origin at the centre u_5 of the square mesh.

We note that the given poisson partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ is symmetrical about x and y axes and also about the line $y=x$.

Hence we have $u_1 = u_3 = u_7 = u_9$ and $u_2 = u_4 = u_6 = u_8$

Hence we have to find u_1, u_2, u_5 only.

The standard five point formula for the given poisson equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \quad \text{—} (1)$$

using (1) at u_7 $(i=-1, j=-1)$ we have

$$u_{-2,-1} + u_{0,-1} + u_{-1,-2} + u_{-1,0} - 4u_{-1,-1} = 8(-1)^2(-1)^2$$

(ie) $0 + u_8 + 0 + u_4 - 4u_7 = 8$

(ie) $u_2 + u_2 - 4u_1 = 8$

(ie) $u_2 - 2u_1 = 4 \quad \text{—} (2)$

using (1) at u_2 $(i=0, j=1)$ we have

$$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8 \times 0 \times 1 = 0$$

(ie) $u_1 + u_3 + u_5 + 0 - 4u_2 = 0$

$\therefore 2u_1 - 4u_2 + u_5 = 0$ —— (3)

using (1) at u_5 $(i=0, j=0)$ we have

$$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 0$$

$\therefore u_4 + u_6 + u_8 + u_2 - 4u_5 = 0$

$4u_2 - 4u_5 = 0$

$u_2 = u_5$ —— (4)

Solving these equations (2), (3), (4) we get

$u_1 = -3, \quad u_2 = -2, \quad u_5 = -2$

$\therefore$ The solution to the given poission equation at the 9 interior mesh points are

$u_1 = u_3 = u_7 = u_9 = -3$

$u_2 = u_4 = u_6 = u_8 = -2$ and $u_5 = -2$.