Unit - I ①

Partial Differential Equations

Formation of Partial Differential equations by Elimination of arbitrary constants

Consider an equation $f(x, y, z, a, b) = 0$ —①

where a & b denote arbitrary constants.

A p.d.e is formed by eliminating the arbitrary constants that occur in the functional relation between the variables.

Using $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

1. Form the p.d.e by eliminating the arbitrary constants a & b form $z = ax + by$ —①

Diff p.w.r.to x we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a.$$

Diff p.w.r.to y we get

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

∴ Eqn ① becomes, $z = px + qy$.

2. Eliminate the arbitrary constants a & b form $z = (x^2+a)(y^2+b)$.

Sol:

$$z = (x^2+a)(y^2+b)$$

Diff p. w.r. to x.

$$p = \frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow \frac{p}{2x} = y^2+b$$

Diff p.w.r.to y.

$$q = 2y(x^2+a) \Rightarrow \frac{q}{2y} = x^2+a$$

$\therefore$ Eqn (1) becomes

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$4xyz = pq.$$

3. $z = a(x+y)+b$.

Sol:

$$z = a(x+y)+b$$

Diff w.r.to x.

$$p = a \rightarrow (1)$$

Diff w.r.to y.

$$q = a \rightarrow (2)$$

From (1) & (2) we get $\cancel{pq = a}$ $p = q$.

4. Find the partial differential equation of all planes having equal intercepts on the x & y axis

Sol: Intercept form of the plane equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Given $a=b$ (Equal intercepts on the x & y axis)

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1.$$

Diff p.w.r.to x, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{1}{a}=-\frac{1}{c}\frac{\partial z}{\partial x}=-\frac{1}{c}p \rightarrow (1)$$

Diff p.w.r. to y, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{1}{a}=-\frac{1}{c}\frac{\partial z}{\partial y}=-\frac{1}{c}q \rightarrow (2)$$

From (1) & (2),

$$-\frac{1}{c}p=-\frac{1}{c}q$$

$$p=q.$$

6. $(x-a)^2+(y-b)^2+z^2=1.$

Sol: Diff w.r.to x, we get

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$\cancel{2}(x-a) = -\cancel{2}zp$$

$$x-a = -zp \rightarrow (1)$$

Diff p.w.r.to y, we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$\cancel{2}(y-b) = -\cancel{2}zq$$

$$y-b = -zq \rightarrow (2)$$

Using (1) & (2), we get

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2p^2 + z^2q^2 + z^2 = 1$$

$$z^2(p^2+q^2+1) = 1.$$

(6) $z = (x+a)^3 + (y-b)^2$

Sol:

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = 3(x+a)^2 \cdot 1$$

$$\Rightarrow \frac{p}{3} = (x+a)^2$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = 2(y-b)$$

$$\frac{q}{2} = y-b.$$

$$z = \left(\frac{p}{3}\right)^{3/2} + \left(\frac{q}{2}\right)^2 \quad z = \left(\frac{p}{3}\sqrt{\frac{p}{3}}\right)^3 + \left(\frac{q}{2}\right)^2$$

$$z = \left(\frac{p^{3/2}}{3^{3/2}}\right)^3 + \left(\frac{q}{2}\right)^2.$$

(7) $(x-a)^2+(y-b)^2 = z^2 \cot^2\alpha$.

Sol:

$$(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$$

Diff w.r. to x we get

$$\cancel{2}(x-a) = \cancel{2} z p \cot^2\alpha$$

$$x-a = zp \cot^2\alpha$$

Diff w.r. to y, we get

$$\cancel{2}(y-b) = \cancel{2} z q \cot^2\alpha$$

$$y-b = zq \cot^2\alpha$$

$$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$$

$\div z^2\cot^2\alpha$, $\quad p^2\cot^2\alpha + q^2\cot^2\alpha = 1$.

Formation of PDE by Elimination of arbitrary constants Functions:

Formation of PDE by elimination of arbitrary functions from the elimination of one arbitrary functions from a given relation gives a PDE of first order while elimination of two arbitrary function from a given relation gives a second order or higher order PDE.

Using $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$,

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

$$\phi(u,v)=0 \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

1. Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ form a PDE.

Sol:

Given $z = f\left(\frac{y}{x}\right)$

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2}$$

$$\Rightarrow p = f'\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2} \rightarrow (1)$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow q = f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} \rightarrow (2)$$

from (1) & (2).

$$\frac{p}{q} = \frac{-y}{x^2}\cdot\frac{x}{1}$$

$$\frac{p}{q} = \frac{-y}{x}$$

$$px = -qy$$

$$\Rightarrow px + qy = 0.$$

(2) $z = f(xy)$

Sol: $z = f(xy)$

Diff w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'(xy) \cdot y$$

$$p = f'(xy) \cdot y$$

Diff w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'(xy) \cdot x$$

$$q = f'(xy) \cdot x$$

$$\frac{p}{q} = \frac{y}{x}$$

$$px = qy$$

$$\Rightarrow px - qy = 0.$$

(3) $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0.$

Sol:

Here $u = z^2 - xy$, $v = \frac{x}{z}$

$$\frac{\partial u}{\partial x} = 2z \frac{\partial z}{\partial x} - y$$

$$= 2zp - y$$

$$\frac{\partial u}{\partial y} = 2z \frac{\partial z}{\partial y} - x$$

$$= 2zq - x$$

$$\frac{\partial v}{\partial x} = \frac{z \cdot 1 - x \cdot \frac{\partial z}{\partial x}}{z^2}$$

$$= \frac{z - xp}{z^2}$$

$$\frac{\partial v}{\partial y} = \frac{z \cdot 0 - x \cdot q}{z^2} = \frac{-xq}{z^2}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2zp-y & \frac{z-xp}{z^2} \\ 2zq-x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$(2zp-y)\left(-\frac{xq}{z^2}\right) - (2zq-x)\left(\frac{z-xp}{z^2}\right) = 0$$

$$\frac{1}{z^2}\left[-\cancel{2zp}xq + yxq - 2zqz + \cancel{2zq}xp + xz - x^2p\right] = 0$$

$$xyq + xz - 2z^2q - x^2p = 0$$

$$xz = 2z^2q + x^2p - xyq$$

$$xz = x^2p - q(xy - 2z^2)$$

4. $z = f(x^2+y^2)$.

Sol:

$$z = f(x^2+y^2)$$

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

$$p = 2x\, f'(x^2+y^2)$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y$$

$$q = 2y\, f'(x^2+y^2)$$

$$\frac{p}{q} = \frac{\cancel{2}x\, \cancel{f'(x^2+y^2)}}{\cancel{2}y\, \cancel{f'(x^2+y^2)}}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py - qx = 0.$$

5. $z = xf(2x+y) + g(2x+y)$

Sol:

$$z = xf(2x+y) + g(2x+y)$$

Diff p.w.r. to x.

$$p = \frac{\partial z}{\partial x} = f(2x+y) + xf'(2x+y)\cdot 2 + g'(2x+y)\cdot 2 \rightarrow (1)$$

$$q = xf'(2x+y) + g'(2x+y) \rightarrow (2)$$

$$r = f'(2x+y)\cdot 2 + xf''(2x+y)\cdot 4 + f'(2x+y)\cdot 2 + g''(2x+y)\cdot 4 \rightarrow (3)$$

$$t = xf''(2x+y) + g''(2x+y) \rightarrow (4)$$

$$s = f'(2x+y) + 2xf''(2x+y) + 2g''(2x+y) \rightarrow (5)$$

$(3) \Rightarrow r = 4f'(2x+y) + 4[xf''(2x+y) + g''(2x+y)]$

$$r = 4f'(2x+y) + 4t \rightarrow (6)$$

$(5) \Rightarrow s = 2[xf''(2x+y) + g''(2x+y)] + f'(2x+y)$

$$s = 2t + f'(2x+y) \rightarrow (7)$$

$(6) - 4\times(7)$

$$r = 4f'(2x+y) + 4t$$

$$4s = 4f'(2x+y) + 8t$$

$(-) \quad (-) \quad (-)$

$$r - 4s = -4t$$

$$r - 4s + 4t = 0.$$

(6) From the p.de by eliminating ϕ from

$\phi(x^2+y^2+z^2, ax+by+cz)=0$.

Sol: Here $u = x^2+y^2+z^2$ $\quad v = ax+by+cz$.

$\frac{\partial u}{\partial x} = 2x + 2zp$ $\quad \frac{\partial v}{\partial x} = a + cp$

$\frac{\partial u}{\partial y} = 2y + 2zq$ $\quad \frac{\partial v}{\partial y} = b + cq$.

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x+2zp & a+cp \\ 2y+2zq & b+cq \end{vmatrix} = 0$$

$2(x+zp)(b+cq) - 2(y+zq)(a+cp) = 0$.

Singular Integrals:

Problems based on solution of P.DE in ordinary cases:

1. Solve: $\frac{\partial z}{\partial x} = \sin x$.

Sol: diff w.r.to x.

Integrating p.w.r.to x.

$z = -\cos x + f(y)$.

$f(y)$ is arbitrary constant.

2. Solve: $\frac{\partial^2 z}{\partial x^2} = xy$.

Sol:
$$\frac{\partial^2 z}{\partial x^2} = xy.$$

ie, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$

Integrating p.w.r.to x,
$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y)$$

Integrating p.w.r.to x
$$z = \frac{x^3}{3 \cdot 2} y + x f(y) + F(y).$$

where both $f(y)$ & $F(y)$ are arbitary.

3. Solve: $\frac{\partial z}{\partial x} = 3x - y$, $\frac{\partial z}{\partial y} = -x + \cos y$.

Sol:
$$\frac{\partial z}{\partial x} = 3x - y.$$

Integrating p.w.r.to x,
$$z = \frac{3x^2}{2} - xy + f(y)$$

$$\frac{\partial z}{\partial y} = -x + \cos y$$

Integrating p.w.r.to y,
$$z = -xy + \sin y + c.$$

$$z = \frac{3}{2}x^2 - xy + \sin y + c.$$

4. Solve: $\frac{\partial^2 z}{\partial x^2} = \sin y$

Sol:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \sin y$$

Integrating p.w.r.to x,

$$\frac{\partial z}{\partial x} = x \sin y + f(y)$$

Again integrating p.w.r. to x

$$z = \frac{x^2}{2} \sin y + x f(y) + F(y)$$

where both $f(y)$ & $F(y)$ are arbitrary.

5. Solve: $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that $x=0$, $\frac{\partial z}{\partial x} = a \sin y$, and $\frac{\partial z}{\partial y} = 0$.

Sol: Given $\frac{\partial^2 z}{\partial x^2} = a^2 z$

$$\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$$

The A.E is $m^2 + a^2 = 0$

$$m^2 = a^2$$

$$m = \pm ia$$

$$z = Ae^{ax} + Be^{-ax}$$

Since z is a function of x & y. Therefore A & B will be the functions of y alone. Hence

$$z = f(y) e^{ax} + \phi(y) e^{-ax} \rightarrow (1)$$

where f(y) & φ(y) are functions of y alone.

Diff w.r.to x,

$$\frac{\partial z}{\partial x} = p = a f(y) e^{ax} - a \phi(y) e^{-ax} \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = q = f'(y) e^{ax} + \phi'(y) e^{-ax} \rightarrow (3)$$

Case: (i) Given $\frac{\partial z}{\partial x} = a \sin y$ when $x = 0$.

$$(2) \Rightarrow a \sin y = a f(y) - a \phi(y)$$

$$a \sin y = a[f(y) - \phi(y)]$$

$$\sin y = f(y) - \phi(y) \rightarrow (4)$$

case: (ii) Given $\frac{\partial z}{\partial y} = 0$ when $x = 0$.

$$(3) \Rightarrow 0 = f'(y) + \phi'(y)$$

Integrating, we get

$$f(y) + \phi(y) = k \rightarrow (5)$$

$$(4) + (5) \Rightarrow f(y) = \frac{1}{2}(\sin y + k)$$

(5) - (4) $\Rightarrow$ $\phi(y) = \frac{1}{2}(K - \sin y)$

$$z = \frac{1}{2}(\sin y + K)e^{ax} + \frac{1}{2}(K - \sin y)e^{-ax}$$

$$= \frac{1}{2}\sin y\, e^{ax} + \frac{1}{2}Ke^{ax} + \frac{1}{2}Ke^{-ax} - \frac{1}{2}\sin y\, e^{-ax}$$

$$= \frac{1}{2}\sin y\,(e^{ax} - e^{-ax}) + \frac{1}{2}K(e^{ax} + e^{-ax})$$

$$= \sin y\left(\frac{e^{ax} - e^{-ax}}{2}\right) + K\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$z = \sin y \sinh ax + K\cosh ax.$$

Define : Singular Integral

Let $f(x, y, z, p, q) = 0$. $\rightarrow$ (1)

Let the complete integral be

$$\phi(x, y, z, a, b) = 0 \rightarrow (2)$$

Diff (2) p.w.r.to a & b in turn we get

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (3) \text{ and}$$

$$\frac{\partial \phi}{\partial b} = 0 \rightarrow (4)$$

The elimination of a & b from the three equations (2), (3) & (4) if it exists, is called the singular integral.

Type: 1 $f(p,q)=0$.

[The equations contain p and q only]

Suppose that $z=ax+by+c$ is a trial solution of $f(p,q)=0$.

where $p=a$, $q=b$ we get $f(a,b)=0$

Here a & b are the constant.

Eliminate any one constant we get the complete solution.

1. Find the complete solution of $\sqrt{p}+\sqrt{q}=1$

Sol:

Given $\sqrt{p}+\sqrt{q}=1 \rightarrow (1)$

This equation of the form $f(p,q)=0$.

Hence the trial solution is $z=ax+by+c \rightarrow (2)$

where $p=a$ & $q=b$.

Substitute in eqn (1) we get

$$\sqrt{a}+\sqrt{b}=1$$

$$\Rightarrow \sqrt{b}=1-\sqrt{a} \Rightarrow b=(1-\sqrt{a})^2$$

$$\therefore z=ax+(1-\sqrt{a})^2y+c.$$

2. $p+q=pq$.

Sol: Given $p+q=pq \rightarrow (1)$

This equation of the form $f(p,q)=0$

Hence the trial solution is $z=ax+by+c \rightarrow (2)$

where $p=a$ & $q=b$

Substitute in eqn ①, we get

$$a+b=ab$$

$$\Rightarrow b-ab=a$$

$$b-ab=a$$

$$\Rightarrow b(1-a)=a$$

$$b=\frac{a}{1-a}$$

The complete solution is $z=ax+\left(\frac{a}{1-a}\right)y+c$

⑧ $p^2+q^2=npq$

Sol: Given $p^2+q^2=npq$

This eqn is of the form $z=ax+f(p,q)=0$

Hence the trial solution is $z=ax+by+c$

where $p=a$ & $q=b$

$$a^2+b^2=nab$$

$$b^2-nab+a^2=0$$

$$b=\frac{na\pm\sqrt{a^2n^2-4a^2}}{2}$$

$$=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]$$

The complete solution is

$$z=ax+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]y+c$$

6

(4) $p - 3q = 6$

Sol: Given $p - 3q = 6$

This eqn of the form $f(p,q) = 0$

Hence the trial solution is $z = ax + by + c$

where $p = a$ & $q = b$

$$a - 3b = 6.$$

$$\Rightarrow -3b = 6 - a$$

$$\Rightarrow b = \frac{6-a}{-3} = -2 + \frac{a}{3}$$

The complete solution is

$$z = ax + \left(-2 + \frac{a}{3}\right)y + c$$

(5) $p - q = 0$.

Sol: Given $p - q = 0$.

This eqn of the form $f(p,q) = 0$

Hence the trial solution is $z = ax + by + c \rightarrow (2)$

Sub. (2) in (1). Here $p = a$ & $q = b$

$$a - b = 0$$

$$b = a.$$

The complete solution is

$$z = ax + ay + c = a(x+y) + c$$

Type : 2 Clairaut's form

$$z = px + qy + f(p,q).$$

This eqn of the form $z = px + qy + f(p,q)$.

$\therefore$ The complete integral is

$$z = ax + by + f(a,b)$$

To find the singular integral

Diff p.w.r. to a & b.

we get the solution in terms of x, y, z.

To find the general solution

put $b = f(a)$

Eliminate 'a' we get the general solution.

1\. Solve : $z = px + qy + pq$.

Sol: Given $z = px + qy + pq \rightarrow (1)$

This eqn is of the form. $z = px + qy + f(p,q) \rightarrow (2)$

$\therefore$ The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by + ab.$$

To find singular integral

Diff p.w.r.to a & b.

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + b = 0$$

$$\Rightarrow b = -x$$

$\frac{\partial z}{\partial b} = 0 \Rightarrow y + a = 0$

$\Rightarrow a = -y.$

$\therefore z = (-y)x + (-x)y + (-y)(-x)$

$= -xy - xy + xy$

$z = -xy$

$z + xy = 0.$

which is a singular solution.

To get the general integral

put $b = f(a)$ in eqn ①.

$z = ax + f(a)y + af(a) \rightarrow ④$

Diff p.w.r.to a, $\frac{\partial z}{\partial a} = 0.$

$\Rightarrow x + f'(a)y + a f'(a) + f(a) = 0 \rightarrow ⑤$

Eliminate 'a' between ④ & ⑤ we get the general solution.

② $z = px + qy + p^2 - q^2$

Sol:

Given $z = px + qy + p^2 - q^2$ — ①

This eqn of the form $z = px + qy + f(p,q)$ — ②

The complete integral is

$z = ax + by + f(a,b)$

(ie) $z = ax + by + a^2 - b^2$ → ③

To find singular integral

Diff p.w.r.to a & b,

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + 2a = 0$

$2a = -x$

$a = \frac{-x}{2}$.

$\frac{\partial z}{\partial b} = 0 \Rightarrow y - 2b = 0$.

$\Rightarrow y = 2b$

$\Rightarrow b = \frac{y}{2}$.

Sub a, b in (2),

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}.$$

$$= \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$$z = \frac{-x^2 + y^2}{4}$$

$4z = y^2 - x^2$ is the singular integral

To find the general integral

Put $b = f(a)$ in (2)

$$z = ax + f(a)y + a^2 - (f(a))^2 \rightarrow (4)$$

$\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + f'(a)y + 2a - 2f(a).f'(a) = 0 \rightarrow (5)$

Eliminate 'a' between (4) & (5) we get

the general solution.

(8) Solve: $z = px + qy + \sqrt{p^2+q^2+1}$ (b)

Sol:

Given $z = px + qy + \sqrt{p^2+q^2+1}$

This eqn is of the form $z = px + qy + f(p,q)$

$\therefore$ The complete integral is

$z = ax + by + f(a,b)$

(ie) $z = ax + by + \sqrt{a^2+b^2+1} \longrightarrow (1)$

To find singular integral

Diff p.w.r.to a & b,

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + \frac{1}{2}(a^2+b^2+1)^{-\frac{1}{2}} \cdot 2a = 0$

$\Rightarrow x + \frac{a}{\sqrt{a^2+b^2+1}} = 0$

ie. $x = \frac{-a}{\sqrt{a^2+b^2+1}} \longrightarrow (2)$

$\frac{\partial z}{\partial b} = 0 \Rightarrow y + \frac{1}{2}(a^2+b^2+1)^{-\frac{1}{2}} \cdot 2b = 0$

$\Rightarrow y + \frac{b}{\sqrt{a^2+b^2+1}} = 0$

$\Rightarrow y = \frac{-b}{\sqrt{a^2+b^2+1}} \longrightarrow (3)$

$x^2 + y^2 = \frac{a^2}{a^2+b^2+1} + \frac{b^2}{a^2+b^2+1}$

$= \frac{a^2+b^2}{a^2+b^2+1}$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{+1}{a^2+b^2+1}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{a^2+b^2+1}}$$

$$\sqrt{a^2+b^2+1} = \frac{1}{\sqrt{1-x^2-y^2}}$$

(2) $\Rightarrow$ $x = -a\sqrt{1-x^2-y^2} \Rightarrow a = \frac{-x}{\sqrt{1-x^2-y^2}}$

(3) $\Rightarrow$ $y = -b\sqrt{1-x^2-y^2} \Rightarrow b = \frac{-y}{\sqrt{1-x^2-y^2}}$

Sub in (1)

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2=1$ is the singular solution.

To get the general integral

Put $b = f(a)$ in (1).

$$z = ax + f(a)y + \sqrt{1+a^2+(f(a))^2} \quad \text{--- (4)}$$

Diff (4) p.w.r.to 'a',

$$0 = x + f'(a)y + \frac{1}{2}(1+a^2+(f(a))^2)^{-\frac{1}{2}} \cdot (2a + 2f(a) \cdot f'(a))$$

$$0 = x + f'(a)y + \frac{a + f(a)f'(a)}{\sqrt{1+a^2+(f(a))^2}} \rightarrow (5)$$

Eliminate 'a' between (4) & (5) we get the general solution.

(4) $z = px + qy - 2\sqrt{pq}$

Sol:

This eqn is of the form

$$z = px + qy + f(p, q)$$

The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by - 2\sqrt{ab} \quad \text{--- (1)}$$

To find singular integral

Diff p.w.r.to a & b in (1)

$$\frac{\partial z}{\partial a} = 0$$

$$\Rightarrow x + 0 - 2\frac{1}{2}(ab)^{-\frac{1}{2}} b = 0$$

$$\Rightarrow x = (ab)^{-\frac{1}{2}} \cdot b$$

$\frac{\partial z}{\partial b} = 0$

$\Rightarrow y - 2 \cdot \frac{1}{2} (ab)^{-\frac{1}{2}} \cdot a = 0$

$\Rightarrow y = (ab)^{-\frac{1}{2}} a$

$\Rightarrow xy = (ab)^{-\frac{1}{2}} \cdot (ab)^{-\frac{1}{2}} \cdot ab$

$= a^{-\frac{1}{2}} b^{-\frac{1}{2}} a^{-\frac{1}{2}} \cdot b^{-\frac{1}{2}} a \cdot b$

$= a^{-1} \cdot b^{-1} \cdot a \cdot b$

$\therefore xy = 1$

Type : 3 $F(z, p, q) = 0$.

This eqn is of the form $f(z, p, q) = 0$ —(1)

Let $z = f(x+ay)$ be the solution of (1)

put $x + ay = u$ —(2)

Then $z = f(u)$ —(3)

Substitute $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$ Then

Integrating we get the solution.

1\. Solve: $p(1+q) = qz$.

Sol:

Given $p(1+q) = qz$ —(1)

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = a$

7

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$① \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}$$

$$\therefore 1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\frac{du}{dz} = \frac{a}{az-1}$$

$$du = \frac{a}{az-1}dz$$

Integrating on both sides, we get

$$u = \log(az-1) + b.$$

$$x + ay = \log(az-1).$$

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(2) $z^2 = 1 + p^2 + q^2$.

Sol:

Given $z^2 = 1 + p^2 + q^2$ — ①

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du} \quad q = a\frac{dz}{du}$$

$$① \Rightarrow z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (1+a^2) = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{a^2+1}$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z^2-1}{a^2+1}}$$

$$\Rightarrow \sqrt{a^2+1}\,dz = \sqrt{z^2-1}\,du$$

$$\Rightarrow \sqrt{a^2+1}\,\frac{dz}{\sqrt{z^2-1}} = du$$

Integrating,

$$\sqrt{a^2+1}\int \frac{dz}{\sqrt{z^2-1}} = \int du + b$$

$$\sqrt{a^2+1}\cosh^{-1}(z) = u + b$$

$$\sqrt{a^2+1}\cosh^{-1}(z) = x + ay + b$$

is the complete solution.

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③ $p + q = z$.

Sol:

Given $p + q = z \rightarrow ①$

Let $u = x + ay$

$$p = \frac{dz}{du} \quad q = a\frac{dz}{du}$$

$$\frac{dz}{du} + a\frac{dz}{du} = z$$

$$\frac{dz}{du}(1+a) = z$$

$$\frac{dz}{du} = \frac{z}{1+a}$$

$$(1+a)\frac{dz}{z} = du$$

Integrating,

$$(1+a)\int\frac{dz}{z} = \int du$$

$$(1+a)\log z = u+b$$

$(1+a)\log z = x+ay+b$. is the complete solution.

④ Solve: $p(1-q^2) = q(1-z)$

Sol.

Gn $\quad p(1-q^2) = q(1-z) \rightarrow (1)$

Let $u = x+ay$

$$p = \frac{dz}{du},\ q = a\frac{dz}{du}$$

$$\frac{dz}{du}\left(1-a^2\left(\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(1-z)$$

$$1-a^2\left(\frac{dz}{du}\right)^2 = a\cancel{\frac{dz}{du}}(1-z).$$

$$= a-az$$

$$a^2\left(\frac{dz}{du}\right)^2 = -a+az+1$$

$$a\frac{dz}{du} = \sqrt{1-a+az}$$

$$\frac{a}{\sqrt{1-a+az}}\,dz = du$$

$$a\,(1-a+az)^{-\frac{1}{2}}\,dz = du$$

Integrating we get

$$a\,\frac{(1-a+az)^{\frac{1}{2}}}{\frac{1}{2}\,a} = u+b$$

$2\,(1-a+az)^{\frac{1}{2}} = x+ay+b$ is the complete solution.

Type: 3 (a)

Equation containing (x, p, q) only

i) Let $q = a$

ii) Find p

iii) $dz = p\,dx + q\,dy$

iv) Integrate, we get the complete solution.

① Solve : $p = 2qx$

Sol: Gn $p = 2qx$

Let $q = a \Rightarrow p = 2ax$

$$dz = p\,dx + q\,dy$$

$$dz = 2ax\,dx + a\,dy$$

Integrate,

$$z = \cancel{2}a\frac{x^2}{\cancel{2}} + ay + b$$

$z = ax^2 + ay + b$ is the complete solution

Diff p.w.r. to b,

$0 = 1$ is absurd

There is no singular solution.

(2) solve: $q = 2px$

Sol:

$q = 2px$

Let $q = a$

$a = 2px$

$\Rightarrow p = \frac{a}{2x}$

$dz = p\,dx + q\,dy$

$dz = \frac{a}{2x}dx + a\,dy$

Integrate,

$z = \frac{a}{2}\log x + ay + b$ is the complete solution

Diff p.w.r. to b,

$0 = 1$ is absurd.

There is no singular solution.

Type: 3 (c)

Equation containing y, p, q only

i) Let $p = a$

ii) find q

iii) $dz = p\,dx + q\,dy$

iv) Integrate we get the complete solution.

(1) Solve: $2yp^2 = q$

Sol:

$$2yp^2 = q$$

Let $p = a$

$$2ya^2 = q$$

$$q = 2ya^2$$

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + 2ya^2\,dy$$

Integrate,

$$z = ax + 2a^2\frac{y^2}{2} + b$$

$z = ax + a^2y^2 + b$ is the complete solution.

2. Solve: $q = py + p^2$

Sol:

$$q = py + p^2$$

Let $p = a$

$$q = ay + a^2$$

$$dz = p\,dx + q\,dy$$

$$= a\,dx + (ay + a^2)\,dy$$

Integrate. $z = ax + \frac{ay^2}{2} + a^2y + b$ is the complete solution.

Diff p.w.r.to b, $0 = 1$ is absurd

There is no singular solution

Type: (iv)

Equation containing x, y, p, q.

i) Attach x & p in one side
ii) Attach y & q in other side
iii) Let it be equal to k
iv) Find p & q
v) $dz = p\,dx + q\,dy$
vi) Integrate we get the complete solution

① Solve: $p+q = x+y$

Sol: Gn

$$p - x = y - q = k$$

$$p - x = k, \quad y - q = k$$

$$p = x+k \quad q = y-k$$

$$dz = p\,dx + q\,dy$$

$$dz = (x+k)dx + (y-k)dy$$

Integrating,

$$z = \frac{x^2}{2} + kx + \frac{y^2}{2} - ky + b$$ is the complete solution

Diff p.w.r. to b, $0 = 1$ is absurd

There is no singular solution

② Solve: $pq = xy$

Sol:

$$pq = xy$$

$$\frac{p}{x} = \frac{y}{q} = k \text{ (say)}$$

$$\frac{p}{x} = k, \quad \frac{y}{q} = k.$$

$$p = xk \qquad q = \frac{y}{k}$$

$$dz = p\,dx + q\,dy$$

$$= (xk)dx + \left(\frac{y}{k}\right)dy$$

Integrating we get

$$z = \frac{x^2}{2}k + \frac{y^2}{2k} + b$$ is the complete solution.

Diff p.w.r.to b, $0 = 1$ is absurd

There is no singular integral.

③ Solve: $p^2 y(1+x^2) = qx^2$

Sol: Gn $p^2 y(1+x^2) = qx^2$

$$\Rightarrow \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = k$$

$$\frac{p^2(1+x^2)}{x^2} = k \qquad \frac{q}{y} = k$$

$$p^2 = \frac{kx^2}{1+x^2} \qquad q = yk.$$

$$p = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}\,dx + yk\,dy$$

Integrate.

$$z = \sqrt{k}\int \frac{x}{\sqrt{1+x^2}}\,dx + k\int y\,dy + b$$

$z = \sqrt{k}\ \sqrt{1+x^2} + k\dfrac{y^2}{2} + b$ is the complete solution.

④ $\sqrt{p} + \sqrt{q} = x + y$

Sol:

$$\sqrt{p} + \sqrt{q} = x + y$$

$$\sqrt{p} - x = y - \sqrt{q} = k$$

$$\sqrt{p} - x = k \qquad y - \sqrt{q} = k$$

$$\sqrt{p} = x + k \qquad \sqrt{q} = y - k$$

$$p = (x+k)^2 \qquad q = (y-k)^2$$

$$dz = p\,dx + q\,dy$$

$$dz = (x+k)^2\,dx + (y-k)^2\,dy$$

Integrate, $z = \dfrac{(x+k)^3}{3} + \dfrac{(y-k)^3}{3} + b$ is the complete solution.

Type - V

Form the equation of the type of

$$f(x^m p,\ y^n q,\ z) = 0$$

Case: (i) $m \neq 1,\ n \neq 1$ then $X = x^{1-m}$, $Y = y^{1-n}$

Case: (ii) $m = n = 1$ then put $X = \log x$
$Y = \log y$

Next we follow type (3)

(c) Solve: $p^2+x^2y^2q^2=x^2z^2$

Sol:

$$p^2+x^2y^2q^2=x^2z^2$$

$\div x^2$, $$\frac{p^2}{x^2}+y^2q^2=z^2$$

$$x^{-2}p^2+y^2q^2=z^2$$

$$(x^{-1}p)^2+(yq)^2=z^2 \rightarrow (1)$$

This is of the form $f(x^m p, y^n q, z)=0$

Here $m=-1$, $n=1$

put $X=x^{1-m}$ $\quad Y=y^{1-n}\log y$

$X=x^{1+1}$ $\quad Y=y^{1-1}\log y$

$X=x^2$ $\quad Y=\log y$

$\frac{\partial X}{\partial x}=2x$ $\quad Q=\frac{\partial z}{\partial Y}$

$p=\frac{\partial z}{\partial x}$ $\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$ $\quad q=Q\frac{1}{y}$

$\quad yq=Q$

$p=2x\cdot P$

$\frac{p}{2x}=P$

$x^{-1}p=2P$

$(1) \Rightarrow (2P)^2+Q^2=z^2 \rightarrow (2)$

This is of the form $f(P,Q,z)=0$

we use Type (3)

Let $u = x + ay$

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$

$P = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$

$P = \frac{dz}{du}$

$Q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

$Q = a \frac{dz}{du}$

$(2) \Rightarrow \left(2 \frac{dz}{du}\right)^2 + \left(a \frac{dz}{du}\right)^2 = z^2$

$(4 + a^2) \left(\frac{dz}{du}\right)^2 = z^2$

$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4 + a^2}$

$\frac{dz}{du} = \frac{z}{\sqrt{4 + a^2}}$

$\frac{dz}{z} = \frac{1}{\sqrt{4 + a^2}} \, du$

$\int \frac{dz}{z} = \int \frac{1}{\sqrt{4 + a^2}} \, du$

$\log z = \frac{1}{\sqrt{4 + a^2}} \, u + b$

$\log z = \frac{1}{\sqrt{4 + a^2}} (x + ay) + b$

$\log z = \frac{1}{\sqrt{4 + a^2}} (x^2 + a \log y) + b$

2. Solve: $x^2p^2+y^2q^2=z^2$

Sol:

$$x^2p^2+y^2q^2=z^2$$

$$(xp)^2+(yq)^2=z^2 \rightarrow (1)$$

This eqn is of the form $f(x^m p, y^n q, z)=0$

Here $m=1$, $n=1$.

Put $X=\log x$ $Y=\log y$

$$\frac{\partial X}{\partial x}=\frac{1}{x} \qquad \frac{\partial Y}{\partial y}=\frac{1}{y}$$

$$P=\frac{\partial z}{\partial X} \qquad Q=\frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$p=P\cdot\frac{1}{x} \qquad q=Q\cdot\frac{1}{y}$$

$$xp=P \qquad yq=Q$$

Sub in eqn (1), we get

$$P^2+Q^2=z^2 \rightarrow (2)$$

This eqn is of the form $f(P,Q,z)=0$

Let $u=X+aY$

$$\frac{\partial u}{\partial X}=1 \qquad \frac{\partial u}{\partial Y}=a$$

$$P=\frac{\partial u}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x} \qquad Q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

$$P=\frac{dz}{du} \qquad Q=a\frac{dz}{du}$$

(10)

Sub in (2) we get

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2 \rightarrow (3)$$

$$\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{1+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z}dz = \frac{1}{\sqrt{1+a^2}}du$$

$$\int\frac{dz}{z} = \int\frac{1}{\sqrt{1+a^2}}du$$

$$\log z = \frac{1}{\sqrt{1+a^2}}u + c$$

$$\log z = \frac{1}{\sqrt{1+a^2}}(x+ay)+c$$

$$\log z = \frac{1}{\sqrt{1+a^2}}(\log x + a\log y)+c$$

which is the complete solution.

Type: 6

Eqn of the type $f(z^m p, z^m q) = 0 \rightarrow (1)$

& $f_1(x, z^m p) = f_2(y, z^m q) \rightarrow (2)$

case: (i) if $m \neq -1$ put $z = z^{m+1} \Rightarrow$ Type (1) $\Rightarrow$ Type (4)

case: (ii) if $m = -1$ then put $x = \log z$

$\Rightarrow$ Type (1) $\Rightarrow$ Type (4)

① Solve $z^2(p^2+q^2) = x^2+y^2$

Sol: Given $z^2(p^2+q^2) = x^2+y^2$

$$(zp)^2+(zq)^2 = x^2+y^2 \rightarrow ①$$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^m q)$$

Here $m \neq -1$,

Put $Z = z^{m+1}$

$$\Rightarrow Z = z^{1+1} = z^2.$$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$P = 2zp$$

$$\frac{P}{2} = zp$$

Similarly, $\frac{Q}{2} = zq$

Sub in eqn ①, we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2+y^2$$

$$P^2+Q^2 = 4(x^2+y^2)$$

$$P^2-4x^2 = -Q^2+4y^2.$$

This eqn is of the form $f_1(x,P) = f_2(y,Q)$

$$\therefore P^2-4x^2 = 4y^2-Q^2 = 4a^2$$

$$P^2 = 4a^2+4x^2 \qquad Q^2 = -4a^2+4y^2$$

$$P = 2\sqrt{a^2+x^2} \qquad Q = 2\sqrt{y^2-a^2}$$

$$dz = P\,dx + Q\,dy$$

$$dz = 2\sqrt{a^2+x^2}\,dx + 2\sqrt{y^2-a^2}\,dy$$

$$\int dz = 2\int\sqrt{a^2+x^2}\,dx + 2\int\sqrt{y^2-a^2}\,dy$$

$$z = 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$

$$z^2 = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right) + b$$

$$= x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\left[\sinh^{-1}\left(\frac{x}{a}\right) - \cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$

2. Solve: $p^2 + q^2 = z^2(x^2+y^2)$

Sol: $p^2+q^2 = z^2(x^2+y^2) \to (1)$

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2+y^2$$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^n q)$$

Here $m = -1$.

put $Z = \log z$.

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$P = \frac{1}{z}\cdot p$$

Similarly, $Q = \frac{1}{z}\cdot q$

Sub in eqn (1), we get

$$p^2 - x^2 = y^2 - Q^2$$

This eqn is of the form

$$f_1(x, p) = f_2(y, Q) \quad \text{Type (4)}$$

$$p^2 - x^2 = y^2 - Q^2 = a^2$$

$$p^2 - x^2 = a^2 \qquad y^2 - Q^2 = a^2$$

$$p^2 = x^2 + a^2 \qquad Q^2 = y^2 - a^2$$

$$p = \sqrt{x^2 + a^2} \qquad Q = \sqrt{y^2 - a^2}$$

$$\therefore dz = p\,dx + Q\,dy$$

$$dz = \sqrt{x^2+a^2}\,dx + \sqrt{y^2-a^2}\,dy$$

$$\int dz = \int \sqrt{x^2+a^2}\,dx + \int \sqrt{y^2-a^2}\,dy$$

$$z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b.$$

$(\because Z = \log z)$

$$\log z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b.$$

(ii) Lagrange's Linear Equation:

Method of Grouping:

In the auxilary equation

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ if the variables can be sepoated in any pair of equations, then we get a solution of the form

$u(x,y) = a$ & $v(x,y) = b$.

(1) Solve: $px + qy = z$

Sol: Given $px + qy = z$

$P = x,\ Q = y,\ R = z$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$x = (y\,c_1)$$

$$\Rightarrow \frac{x}{y} = c_1 \quad \text{(ie)}\ u = \frac{x}{y}$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$$y = zc_2$$

$$\Rightarrow \frac{y}{z} = c_2 \quad \text{(ie)}\ v = \frac{y}{z}$$

The solution of the given p.d.e is

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

(2) Write the general integral of

$$pyz + qzx = xy.$$

Sol: The given eqn is of the form

$$Pp + Qq = R.$$

$$P = yz, \quad Q = zx, \quad R = xy$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$

$$\Rightarrow \int x\,dx = \int y\,dy$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\Rightarrow c_1 = x^2 - y^2$$

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow c_2 = y^2 - z^2$$

$$\therefore \phi(x^2 - y^2, \; y^2 - z^2) = 0.$$

(3) Solve: $x(y-z)p + y(z-x)q = z(x-y)$

Sol: The given eqn is of the form

$$Pp + Qq = R$$

$$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} = \text{Each ratio}$$

$$dx+dy+dz=0$$

Integrating, $x+y+z+c_1$

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $\log x + \log y + \log z = \log c_2$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\therefore \phi(x+y+z,\ xyz) = 0.$$

(4) $(x^2-yz)p + (y^2-zx)q = z^2-xy$.

Sol: This eqn is of the form $Pp + Qq = R$

$P = x^2-yz \quad Q = y^2-zx \quad R = z^2-xy$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xyz+y^3-xyz+z^3-xyz}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)(dx+dy+dz)$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c$$

$$x^2+y^2+z^2 = (x+y+z)^2 + c$$

$$x^2+y^2+z^2 = x^2+y^2+z^2+2xy+2yz+2zx+c$$

$$-2(xy+yz+zx) = c$$

$$xy+yz+zx = u \text{ (constant)}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\frac{d(x-y)}{(x^2-y^2)+z(x-y)} = \frac{d(y-z)}{y^2-z^2+x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(y+z+x)}$$

12

Integrating on both sides, we get

$$\log(x-y) = \log(y-z) + \log c$$

$$\log(x-y) - \log(y-z) = \log c$$

$$\log \frac{x-y}{y-z} = \log c$$

$$\frac{x-y}{y-z} = c.$$

The general solution is

$$\phi\left(xy+yz+zx, \frac{x-y}{y-z}\right) = 0.$$

Method of Multipliers:

choose any three multipliers l, m, n which may be constants or functions of x, y, z, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

It is possible to choose l, m, n such that

$lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on integration we get a solution $u = a$.

The multipliers l, m, n are called Lagrangian multipliers.

(1) Solve: $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

Sol:

Given $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

This is of the form

$$Pp+Qq=R.$$

where $P=x(y^2-z^2)$ $Q=y(z^2-x^2)$ $R=z(x^2-y^2)$

The subsidiary eqns are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$$

$$\text{Each ratio}=\frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c.$$

$$\text{Each ratio}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log c_1$$

$xyz = c_1$.

$\therefore \phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ xyz\right) = 0.$

2. Solve: $(mz-ny)p + (nx-lz)q = ly-mx$

Sol:

Given $(mz-ny)p+(nx-lz)q = ly-mx$

This eqn is of the form $Pp+Qq=R$

where $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

$$\text{Each ratio} = \frac{dx+dy+dz}{mz-ny+nx-lz+ly-mx}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{xmz-xny+nyx-ylz+lyz-mxz}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz = 0.$$

Integrating.

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c.$$

$$\text{Each ratio} = \frac{l\,dx+m\,dy+n\,dz}{lmz-lny+mnx-mlz+lny-mnx}$$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

Integrating

$$lx + my + nz = c_1$$

The general solution is

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ lx+my+nz\right) = 0.$$

(8) Solve: $(3z-4y)p + (4x-2z)q = 2y-3x$

Sol:

This eqn is of the form $Pp + Qq = R$

$P = 3z-4y$, $Q = 4x-2z$, $R = 2y-3x$

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{3xz - 4yx + 4xy - 2yz + 2yz - 3xz}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\text{Each ratio} = \frac{2\,dx + 3\,dy + 4\,dz}{6z - 8y + 12x - 6z + 8y - 12x}$$

$$\Rightarrow 2\,dx + 3\,dy + 4\,dz = 0.$$

$2x+3y+4z=c_2.$

The general solution is

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ 2x+3y+4z\right)=0$$

(4) Solve: $(y-xz)p+(yz-x)q=(x+y)(x-y)$

Sol:

This eqn is of the form $Pp+Qq=R$

$P=y-xz,\quad Q=yz-x,\quad R=(x+y)(x-y)$

$$\frac{dx}{y-xz}=\frac{dy}{yz-x}=\frac{dz}{(x+y)(x-y)}=\frac{dz}{x^2-y^2}$$

$$\text{Each ratio}=\frac{x\,dx+y\,dy+z\,dz}{xy-x^2z+y^2z-xy+x^2z-zy^2}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$

Integrating

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$$

$$\Rightarrow u=\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}$$

$$\text{Each ratio}=\frac{y\,dx+x\,dy+dz}{y^2-xyz+xyz-x^2+x^2-y^2}$$

$$y\,dx+x\,dy+dz=0.$$

Integrating, $\cancel{yx}+\cancel{xy}+\cancel{z}=$