

UNIT II - FOURIER SERIES

Definition:

Periodic function:

A function $f(x)$ is said to have a period T for all x , $f(x+T) = f(x)$, where T is a +ve constant. The least value of $T > 0$ is called the period of $f(x)$.

e.g: $\sin x$, $\cos x$ are periodic function with period 2π .

Fourier series:

If $f(x)$ is a periodic function and satisfies Dirichlet conditions, then it can be represented by an infinite series called Fourier series.

Uses of Fourier series:

Fourier series are particularly suitable for expansion of periodic functions. we come across many periodic functions in voltage, current flux density; applied force, potential and electromagnetic force in electricity, hence

Fourier series are very useful in electrical engineering problems.

Deduction: Sum of the Fourier Series

Continuous	Discontinuous
End point	Middle point
Substitute the value directly	Average values at end points
	$\frac{L.H.S + R.H.S}{2}$

Problems:

① Sum of the Fourier series for

$$f(x) = \begin{cases} x^2, & -\pi \leq x \leq 0 \\ 0, & 0 \leq x \leq \pi \end{cases} \text{ at } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

sol:

$x = \frac{\pi}{2}$ is a continuous point at $(0, \pi)$

$$\therefore f(x) = 0.$$

$x = -\frac{\pi}{2}$ is a continuous point at $(-\pi, 0)$

$$f(x) = x^2$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = \left(-\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$$

$$\therefore f(x) = \frac{\pi^2}{4}$$

② Sum the Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases} \quad \text{at } x=0.$$

Sol: $x=0$ is a discontinuous and end point.

$$\begin{aligned} \text{Sum of the Fourier series} &= \frac{f(0) + f(2)}{2} \\ &= \frac{0 + 2}{2} = \frac{2}{2} \end{aligned}$$

Sum of the Fourier series } = 1.

③ Sum of the Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ x^2, & \pi < x < 2\pi \end{cases} \quad \text{at } x=\pi.$$

Sol: $x=\pi$ is a discontinuous and middle point.

$$\begin{aligned} \text{Sum of the Fourier series} &= \frac{f(\pi^-) + f(\pi^+)}{2} \\ &= \frac{\pi + \pi^2}{2}. \end{aligned}$$

Dirichlet condition:

- i) $f(x)$ is periodic, single valued and finite
- ii) $f(x)$ has a finite no. of finite discontinuities.
- iii) $f(x)$ has no infinite discontinuities
- iv) $f(x)$ has a finite no. of maxima and minima.

Formula for fourier series in $(0, 2\pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x) dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx dx$$

Problems:

- ① Expand $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2 dx$$

$$= \frac{1}{\pi} \left(\frac{x^3}{3} \right)_0^{2\pi}$$

$$= \frac{8\pi^3}{3}$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2 \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx$$

Here $u = x^2$ $\int dv = \int \cos nx$

$$u_1 = 2x \quad v = \frac{\sin nx}{n}$$

$$u_2 = 2$$

$$v_1 = -\frac{\cos nx}{n^2}$$

$$v_2 = \frac{\sin nx}{n^3}$$

$$a_n = \frac{1}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{4\pi^2 \sin 2n\pi}{n} + \frac{4\pi \cos 2n\pi}{n^2} - \frac{2 \sin 2n\pi}{n^3} \right]$$

(∵ $\sin 2n\pi = 0$
 $\cos 2n\pi = 1$)

$$= \frac{1}{\pi} \left[\frac{4\pi}{n^2} \right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx \, dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2 \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$u = x^2 \quad \int dv = \int \sin nx$$

$$u_1 = 2x \quad v = -\frac{\cos nx}{n}$$

$$u_2 = 2 \quad v_1 = -\frac{\sin nx}{n^2} \quad v_2 = \frac{\cos nx}{n^3}$$

$$b_n = \frac{1}{\pi} \left[\frac{-x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{-4\pi^2 \cos 2n\pi}{n} + \frac{4\pi \sin 2n\pi}{n^2} + \frac{2 \cos 2n\pi}{n^3} \right]$$

$$b_n = \frac{1}{\pi} \left[\frac{-4\pi^2}{n} + \frac{4}{n^2} - \frac{4}{n^3} \right] \Big|_{\pi}^0 = 0$$

$$b_n = -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{8\pi^2}{3 \cdot 2} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n} \sin nx.$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n} \sin nx.$$

Deduction:

Put $x=0$ [end points & discontinuous]

$$\frac{f(0) + f(2\pi)}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{0 + 4\pi^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{6\pi^2 - 4\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3 \cdot 4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

② Expand $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2 dx$$

$$= \frac{1}{\pi} \left[-\frac{(\pi - x)^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[+\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \frac{2\pi^3}{3}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx dx$$

$$u = (\pi - x)^2 \quad \int dv = \int \cos nx$$

$$u_1 = -2(\pi - x)$$

$$v = \frac{\sin nx}{n}$$

$$a_n = \frac{1}{\pi} \left[(\pi-x)^2 \frac{\sin nx}{n} - 2(\pi-x) \cos nx - \frac{2 \sin nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{2(\pi-2\pi) \cos 2n\pi}{n^2} + \frac{2\pi \cos 2n\pi}{n^2} \right]$$

$$= \frac{1}{\pi} \left(\frac{4\pi \cos 2n\pi}{n^2} \right) = \frac{1}{\pi} \left(\frac{4\pi}{n^2} \right)$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} (\pi-x)^2 \sin nx \, dx$$

$u = (\pi-x)^2 \quad dv = \sin nx$
 $u_1 = -2(\pi-x) \quad v = -\frac{\cos nx}{n}$
 $u_2 = 2 \quad v_1 = -\frac{\sin nx}{n^2}; v_2 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi} \left[\frac{(\pi-x)^3 \cos nx}{n} - \frac{2(\pi-x) \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{(\pi-2\pi)^3 \cos 2n\pi}{n} + \frac{2 \cos 2n\pi}{n^3} + \frac{\pi^2 \cos 2n\pi}{n} - \frac{2 \cos 0}{n^3} \right]$$

$$= \frac{1}{\pi} \left[-\frac{\pi^3}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3} \right]$$

$$b_n = 0$$

$$\therefore f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

Deduction:
 Put $x=0$ (end point & discontinuous)

$$\frac{f(0) + f(2\pi)}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{\pi^2 + \pi^2}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\pi^2 - \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{4\pi^2}{3 \cdot 4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Problems on $(0, 2\pi)$:

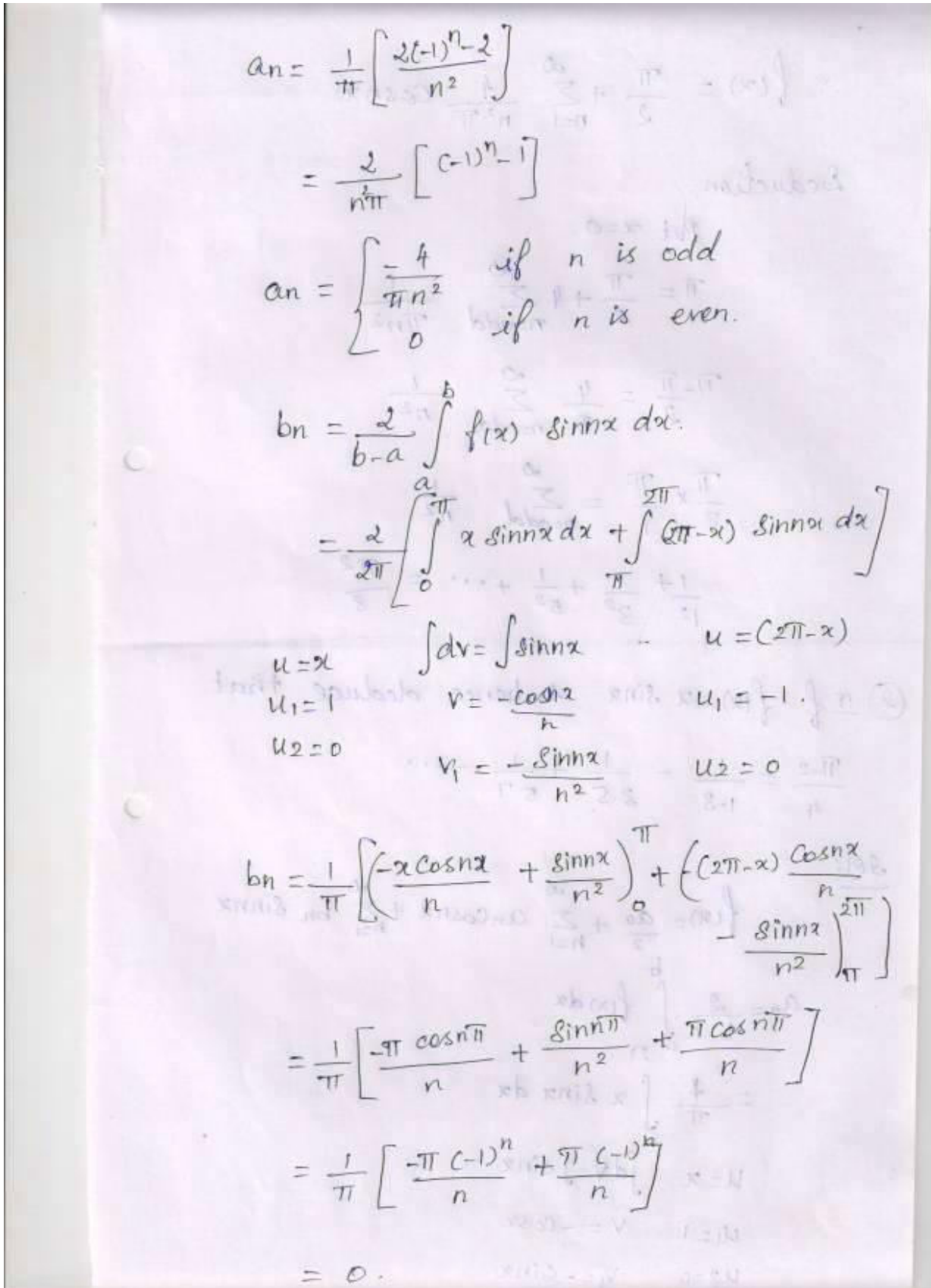
1. Expand $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi < x < 2\pi \end{cases}$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

sol: $a_0 = \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x) dx \right]$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\left(\frac{x^2}{2} \right) \Big|_0^{\pi} + \left(2\pi x - \frac{x^2}{2} \right) \Big|_{\pi}^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2} \right] \\
 &= \frac{1}{\pi} (\pi^2) \\
 a_0 &= \pi. \\
 a_n &= \frac{2}{b-a} \int_a^b f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx \, dx \right] \\
 \begin{array}{l} u = x \\ u_1 = 1 \\ u_2 = 0 \end{array} & \int dv = \int x \cos nx \, dx \quad \begin{array}{l} u = (2\pi - x) \\ u_1 = -1 \\ u_2 = 0 \end{array} \\
 & \rightarrow V = \frac{\sin nx}{n} \\
 & \rightarrow V_1 = -\frac{\cos nx}{n^2}
 \end{array}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \left[\left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_0^{\pi} + \left((2\pi - x) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right) \Big|_{\pi}^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2} \right] \\
 &= \frac{1}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2} \right]
 \end{aligned}$$



$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi} \cos nx$$

Deduction:

Put $x=0$.

$$\pi = \frac{\pi}{2} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi - \pi}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{2} \times \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

② If $f(x) = x \sin x$ hence deduce that

$$\frac{\pi-2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots$$

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x dx$$

$$u = x \quad dv = \sin x$$

$$u_1 = 1 \quad v = -\cos x$$

$$u_2 = 0 \quad v = -\sin x$$

$$a_0 = \frac{1}{\pi} \left[-2 \cos x + \sin x \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-2\pi \cos 2\pi \right]$$

$$= \frac{1}{\pi} \left[-2\pi \right]$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx \sin x dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \left[\sin(n+1)x - \sin(n-1)x \right] dx$$

$u = x$ $dv = \int \sin(n+1)x dx$ $\int dv = \int \sin(n-1)x dx$
 $u_1 = 1$ $v = -\frac{\cos(n+1)x}{n+1}$ $v = -\frac{\cos(n-1)x}{n-1}$
 $V_1 = -\frac{\sin(n+1)x}{(n+1)^2}$ $V_1 = -\frac{\sin(n-1)x}{(n-1)^2}$

$$a_n = \frac{1}{2\pi} \left[-\frac{x \cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{2\pi} - \left[-\frac{x \cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-\frac{2\pi \cos 2(n+1)\pi}{n+1} + \frac{2\pi}{(n+1)^2} \right] - \left[-\frac{2\pi \cos 2(n-1)\pi}{n-1} + \frac{2\pi}{(n-1)^2} \right]$$

$$= \frac{1}{2\pi} \left[\frac{2\pi}{n+1} + \frac{2\pi}{n-1} \right]$$

$$= \frac{2\pi}{2\pi} \left[\frac{-n+1+n+1}{n^2-1} \right]$$

$$= \frac{2}{n^2-1} \quad (n \neq 1)$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \frac{\sin 2x}{2} \, dx$$

$u = x$ $dv = \int \sin 2x \, dx$
 $u_1 = 1$ $V = -\frac{\cos 2x}{2}$
 $u_2 = 0$ $v_1 = -\frac{\sin 2x}{4}$

$$a_1 = \frac{1}{\pi} \left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{2\pi \cos 2\pi}{2} \right]$$

$$= -\frac{1}{2}$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x (\cos(n-1)x - \cos(n+1)x) \, dx$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} x \cos(n-1)x \, dx - \int_0^{2\pi} x \cos(n+1)x \, dx \right]$$

$$= \frac{1}{2\pi} \left[\left(x \frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} \right) - \left(x \frac{\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2} \right) \right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi} \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right]$$

$$b_n = 0.$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin x dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x(1 - \cos 2x) dx$$

$$= \frac{1}{2\pi} \left[x \left(x - \frac{\sin 2x}{2} \right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4} \right]$$

$$= \frac{1}{2\pi} \left[4\pi^2 - \frac{4\pi^2}{2} \right] = \frac{1}{2\pi} \cdot \frac{4\pi^2}{2}$$

$$b_1 = \pi.$$

$$\therefore f(x) = -\frac{1}{2} + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx - \frac{1}{2} \cos x + \pi \sin x$$

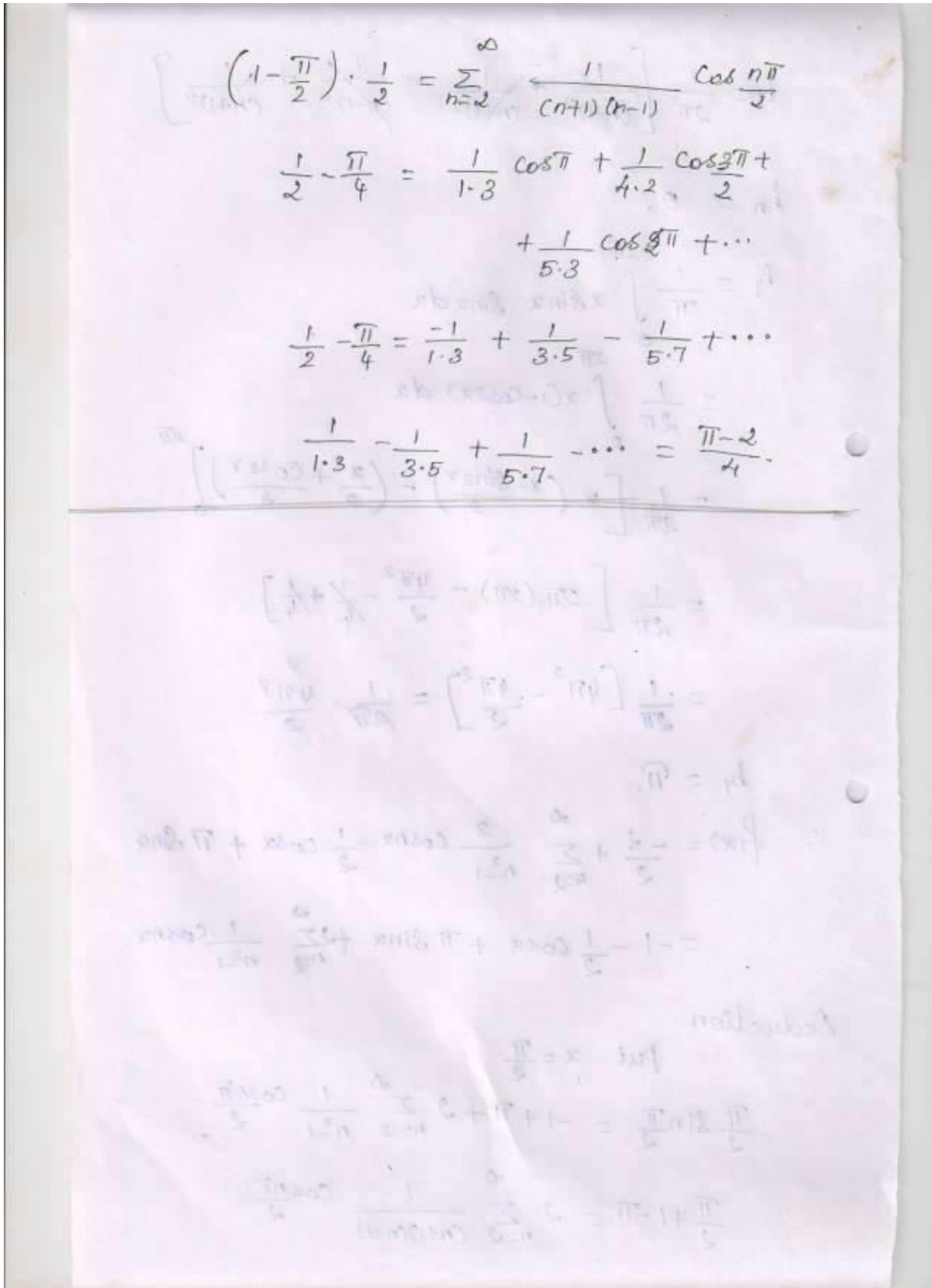
$$= -1 - \frac{1}{2} \cos x + \pi \sin x + 2 \sum_{n=2}^{\infty} \frac{1}{n^2-1} \cos nx.$$

Deduction:

$$\text{put } x = \frac{\pi}{2}$$

$$\frac{\pi}{2} \sin \frac{\pi}{2} = -1 + \pi + 2 \sum_{n=2}^{\infty} \frac{1}{n^2-1} \frac{\cos n\pi}{2}$$

$$\frac{\pi}{2} + 1 - \pi = 2 \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} \frac{\cos n\pi}{2}$$



Formula for Fourier series in $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{l} + \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l}$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x) dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \frac{\cos n\pi x}{l} dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \frac{\sin n\pi x}{l} dx$$

Problems based on $(0, 2l)$:

1. Expand $f(x) = \begin{cases} l-x, & 0 < x \leq l \\ 0, & l \leq x \leq 2l \end{cases}$

hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Sol:

$$a_0 = \frac{2}{2l} \int_0^l (l-x) dx$$

$$= \frac{1}{l} \left(lx - \frac{x^2}{2} \right)_0^l$$

$$= \frac{1}{l} \left(l^2 - \frac{l^2}{2} \right)$$

$$= \frac{l^2}{2l}$$

$$= l$$

$$\begin{aligned}
 a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{n\pi x}{l} dx \\
 &= \frac{2}{2l} \int_0^l x(l-x) \cos \frac{n\pi x}{l} dx \\
 &= \frac{1}{l} \int_0^l (l-x) \cos \frac{n\pi x}{l} dx.
 \end{aligned}$$

$$\begin{aligned}
 u &= l-x & \int dv &= \int \cos \frac{n\pi x}{l} dx \\
 u_1 &= -1 & v &= \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \\
 u_2 &= 0 & & \frac{n\pi}{l}
 \end{aligned}$$

$$\begin{aligned}
 v_1 &= -\frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \left[(l-x) \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l \\
 &= \frac{1}{l} \left[-\frac{\cos \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{\cos 0}{\left(\frac{n\pi}{l}\right)^2} \right] \\
 &= \frac{1}{l} \left[-\frac{(-1)^n \cdot l^2}{n^2 \pi^2} + \frac{l^2}{n^2 \pi^2} \right] \\
 &= \frac{l^2}{l \cdot n^2 \pi^2} [1 - (-1)^n] \\
 a_n &= \begin{cases} \frac{2l}{n^2 \pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}
 \end{aligned}$$

$$b_n = \frac{2}{2l} \int_0^l (l-x) \frac{\sin n\pi x}{l} dx$$

$$u = l-x \quad \int dv = \int \frac{\sin n\pi x}{l} dx$$

$$u_1 = -1 \quad v = -\cos \frac{n\pi x}{l} \Big|_{n\pi/l}$$

$$v_1 = -\frac{\sin n\pi x}{\left(\frac{n\pi}{l}\right)^2}$$

$$b_n = \frac{1}{l} \left[-(l-x) \frac{\cos n\pi x}{\frac{n\pi}{l}} - \frac{\sin n\pi x}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l$$

$$= \frac{1}{l} \left[l \frac{\cos 0}{\frac{n\pi}{l}} \right] = \frac{1}{l} \left[l \cdot \frac{l}{n\pi} \right]$$

$$b_n = \frac{l}{n\pi}$$

$$f(x) = \frac{l}{4} + \sum_{n=\text{odd}}^{\infty} \frac{2l}{n^2\pi^2} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi x}{l}$$

Deduction:

Put $x=l$ (continuous)

$$0 = \frac{l}{4} + \sum_{n=\text{odd}}^{\infty} \frac{2l}{n^2\pi^2} \cos n\pi$$

$$-\frac{l}{4} = \frac{2l}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{1}{4} \cdot \frac{\pi^2}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Put $x = \frac{l}{2}$ (continuous)

$$l - \frac{l}{2} = \frac{l}{4} + \sum_{n=\text{odd}}^{\infty} \frac{2l}{n^2\pi^2} \cos \frac{n\pi l}{2l} + \frac{\pi^2}{4}$$

$$+ \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi \cdot l}{2l}$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{\cos n\pi}{2} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi}{2}$$

$$\frac{l}{4} = \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi}{2}$$

$$\frac{l}{4} \cdot \frac{\pi}{l} = \frac{\sin \pi/2}{1} + \frac{\sin 2\pi/2}{2} + \frac{\sin 3\pi/2}{3} + \frac{\sin 4\pi/2}{4}$$

$$\frac{\pi}{4} = 1 + 0 + \frac{\sin(\pi + \pi/2)}{3} + 0 + \frac{\sin(2\pi + \pi/2)}{5} + \dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{\sin \pi}{5} + \dots$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

$$\textcircled{2} \quad f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$$

sol:

$$2l = 6 \Rightarrow l = \frac{6}{2} = 3$$

$$a_0 = \frac{2}{6-0} \int_0^6 f(x) dx$$

$$\begin{aligned}
 a_0 &= \frac{1}{3} \left[\int_0^3 x dx + \int_3^6 (6-x) dx \right] \\
 &= \frac{1}{3} \left[\left(\frac{x^2}{2} \right)_0^3 + \left(6x - \frac{x^2}{2} \right)_3^6 \right] \\
 &= \frac{1}{3} \left[\frac{9}{2} + 36 - \frac{18}{2} - 18 + \frac{9}{2} \right] \\
 &= \frac{1}{3} [9]
 \end{aligned}$$

$$a_0 = 3$$

$$a_n = \frac{2}{6} \left[\int_0^3 x \cos \frac{n\pi x}{3} dx + \int_3^6 (6-x) \cos \frac{n\pi x}{3} dx \right]$$

$$\begin{aligned}
 &= \frac{1}{3} \int dx = \int \cos \frac{n\pi x}{3} dx \\
 \begin{array}{l} u = x \\ u_1 = 1 \\ u_2 = 0 \end{array} & \begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{l} V = \frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \\ -\frac{\cos \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \end{array} \\
 & \begin{array}{l} u = 6-x \\ u_1 = -1 \\ u_2 = 0 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{1}{3} \left[\left(x \frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\cos \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \right)_0^3 + (6-x) \frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right. \\
 & \quad \left. - \frac{\cos \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \right)_3^6 \\
 &= \frac{1}{3} \left[\frac{(-1)^n 9}{n^2 \pi^2} - \frac{9}{n^2 \pi^2} - \frac{9}{n^2 \pi^2} + \frac{(-1)^n 9}{n^2 \pi^2} \right]
 \end{aligned}$$

$$a_n = \frac{2 \cdot 93}{2 \cdot n^2 \pi^2} [(c-1)^n - 1]$$

$$= \frac{6}{n^2 \pi^2} [(c-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-12}{n^2 \pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \frac{\sin n\pi x}{1} dx$$

$$= \frac{2}{6} \left[\int_0^3 x \frac{\sin n\pi x}{3} dx + \int_3^6 (6-x) \frac{\sin n\pi x}{3} dx \right]$$

$$u = x \quad \int dv = \int \frac{\sin n\pi x}{3} dx$$

$$u_1 = 1 \quad v = \frac{-\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \quad u = 6-x$$

$$u_2 = 0 \quad v_1 = \frac{-\sin \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \quad u_2 = 0$$

$$b_n = \frac{1}{3} \left[\left(-x \frac{\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\sin \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \right) \right]_0^3 +$$

$$\left[(6-x) \frac{\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} - \frac{\sin \frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \right]_3^6$$

$$= \frac{1}{3} \left[\frac{-33(c-1)^n}{\frac{n\pi}{3}} + 3 \frac{(-1)^n \cdot 3}{n\pi} \right]$$

$$= \frac{1}{3} [0]$$

$$b_n = 0.$$

$$\therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{-12}{n^2 \pi^2} \cos \frac{n\pi x}{3}$$

Even function:

If $f(x)$ is an even function, then

$$f(x) = f(-x) \Rightarrow b_n = 0.$$

Ex: $\cos x, |x|, x^2,$

$$f(x) = x^2 \Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$$

$$\therefore f(x) = f(-x).$$

$\therefore f(x)$ is an even function $\Rightarrow b_n = 0.$

Odd function:

If $f(x)$ is an odd function, then

$$-f(x) = +f(-x).$$

Ex: $f(x) = x^3, \sin x, x^3, x \cos x.$

$$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$$

$$\therefore f(-x) = -f(x)$$

$\therefore f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Problems based on $(-\pi, \pi)$ & $(-l, l)$.

First check whether the function is odd

or even.

If the function is even using the fourier
formula in the interval $(-\pi, \pi)$.

$$0 \quad \dots \quad \frac{\infty}{\infty} a_n \cos nx$$

Here $b_n = 0.$

If the function is odd using the fourier formula in the interval $(-\pi, \pi)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx \, dx$$

Here $a_0 = 0$ & $a_n = 0$.

If the function is even using the fourier formula in the interval $(-l, l)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\text{where } a_0 = \frac{2}{b-a} \int_{-l}^l f(x) \, dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{n\pi x}{l} \, dx$$

If the function is odd using the fourier formula in the interval $(-l, l)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{b-a} \int_{-l}^l f(x) \frac{\sin n\pi x}{l} \, dx$$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Sol:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi} & , \quad -\pi \leq -x \leq 0 \\ 1 + \frac{2x}{\pi} & , \quad 0 \leq -x \leq \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi} & , \quad 0 \leq x \leq \pi \\ 1 + \frac{2x}{\pi} & , \quad -\pi \leq x \leq 0 \end{cases}$$

$$f(-x) = f(x)$$

$\therefore f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x - \frac{2x^2}{2\pi} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi - \frac{\pi^2}{\pi} \right] = \frac{2}{\pi} \cdot 0$$

$$a_0 = 0$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx \, dx$$

Here $u = 1 - \frac{2x}{\pi}$ $\int dv = \int \cos nx \, dx$

$$u_1 = -\frac{2}{\pi} \quad v = \frac{\sin nx}{n}$$

$$u_2 = 0 \quad v_1 = -\frac{\cos nx}{n^2}$$

$$a_n = \frac{2}{\pi} \left[\left(1 - \frac{2x}{\pi}\right) \frac{\sin nx}{n} - \frac{2 \cos nx}{\pi n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(1 - \frac{2\pi}{\pi}\right) \frac{\sin n\pi}{n} - \frac{2 \cos n\pi}{\pi n^2} + \frac{2}{\pi n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{-2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2} \right]$$

$$a_n = \frac{2}{\pi n^2} \left[1 - (-1)^n \right]$$

$$a_n = \begin{cases} \frac{8}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=\text{odd}}^{\infty} \frac{8}{\pi n^2} \cos nx$$

$$1 = \frac{8}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^2} \cos nx$$

$$\frac{\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$f(x) = |\cos x|$$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Sol:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi} & , \quad -\pi \leq -x \leq 0 \\ 1 + \frac{2x}{\pi} & , \quad 0 \leq -x \leq \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi} & , \quad 0 \leq x \leq \pi \\ 1 + \frac{2x}{\pi} & , \quad -\pi \leq x \leq 0 \end{cases}$$

$$f(-x) = f(x)$$

$\therefore f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x - \frac{2x^2}{2\pi} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi - \frac{\pi^2}{\pi} \right] = \frac{2}{\pi} \cdot 0 = 0$$

$$a_0 = 0$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

$f(x)$ is even. $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} |\cos x| dx = \frac{2}{\pi} \int_0^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx \right\}$$

$$= \frac{2}{\pi} \left\{ (\sin x) \Big|_0^{\pi/2} - (\sin x) \Big|_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \sin \frac{\pi}{2} - \sin 0 - \sin \pi + \sin \frac{\pi}{2} \right\}$$

$$= \frac{2}{\pi} \left\{ 2 \right\} = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi} \int_{-\pi}^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} \cos x \cos nx dx - \int_{\pi/2}^{\pi} \cos x \cos nx dx \right\}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right) \Big|_0^{\pi/2} \right. \\
 &\quad \left. - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right) \Big|_{\pi/2}^{\pi} \right] \\
 &= \frac{1}{\pi} \left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} \right. \\
 &\quad \left. + \frac{\sin(n-1)\pi/2}{n-1} \right] \\
 &= \frac{1}{\pi} \left[\frac{2 \sin(n+1)\pi/2}{n+1} + \frac{2 \sin(n-1)\pi/2}{n-1} \right] \\
 &= \frac{2}{\pi} \left[\frac{\sin n\pi/2 \cos \pi/2 + \cos \pi/2 \sin n\pi/2}{n+1} \right. \\
 &\quad \left. + \frac{\sin n\pi/2 \cos \pi/2 - \cos \pi/2 \sin n\pi/2}{n-1} \right] \\
 &= \frac{2}{\pi} \left[\frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1} \right] \\
 &= \frac{2}{\pi} \frac{\cos n\pi/2}{n^2-1} [n-1 - n-1] \\
 a_n &= \frac{-4}{\pi} \frac{\cos n\pi/2}{n^2-1} \\
 a_1 &= 2 \left\{ \int_0^{\pi/2} \cos x \cos x dx - \int_{\pi/2}^{\pi} \cos x \cos x dx \right\}
 \end{aligned}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{1 + \cos 2x}{2} \right) dx - \int_{\pi/2}^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$a_1 = \frac{2}{\pi} \left[\left(\frac{1}{2}x + \frac{\sin 2x}{4} \right) \Big|_0^{\pi/2} - \left(\frac{1}{2}x + \frac{\sin 2x}{4} \right) \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{2} \cdot \frac{\pi}{2} + \frac{\sin 2\pi/2}{4} - \frac{1}{2} \cdot \pi - \frac{\sin 2\pi}{4} + \frac{1}{2} \cdot \frac{\pi}{2} + \frac{\sin 2\pi/2}{4} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi}{4} - \frac{\pi}{2} + \frac{\pi}{4} \right]$$

$$= \frac{2}{\pi} \left[\frac{2\pi}{4} - \frac{\pi}{2} \right]$$

$$a_1 = 0.$$

$$f(x) = \frac{4}{\pi \cdot 2} + \sum_{n=2}^{\infty} \frac{-4}{\pi} \cdot \frac{\cos n\pi/2}{n^2 - 1}$$

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{\cos n\pi/2}{n^2 - 1}$$

Problems based on odd function:

1) Determine the Fourier series for the function

$$f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases} \quad \& \text{ hence deduce}$$

$$\text{Given } f(x) = \begin{cases} x-1 & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$$

$$f(-x) = \begin{cases} -x-1 & -\pi < -x < 0 \\ 1-x & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} x+1 & 0 < x < \pi \\ x-1 & -\pi < x < 0 \end{cases}$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function $\Rightarrow a_0 = 0, a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{b_n}{\pi} \sin nx$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x+1) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[-(x+1) \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-(\pi+1) \frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{1}{n} \right]$$

$$= \frac{2}{\pi} \left[-(\pi+1) \frac{(-1)^n}{n} + \frac{1}{n} \right]$$

$$b_n = \frac{2}{n\pi} \left[1 - (\pi+1)(-1)^n \right]$$

2. Prove that $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$
in the interval $(-\pi, \pi)$.

Sol:

$$\text{Let } f(x) = \frac{x(\pi^2 - x^2)}{12}$$

$$f(-x) = (-x) \left(\frac{\pi^2 - (-x)^2}{12} \right)$$

$$= -x \left(\frac{\pi^2 - x^2}{12} \right)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Let the required fourier series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{x(\pi^2 - x^2)}{12} \sin nx \, dx$$

$$u = x\pi^2 - x^3 \quad \int dv = \int \sin nx = \frac{2}{12\pi} \int_0^{\pi} (\pi^2 x^2 - x^3) \sin nx \, dx$$

$$u_1 = \pi^2 - 3x^2 \quad \downarrow \quad v = \frac{-\cos nx}{n}$$

$$u_2 = -6x \quad \downarrow \quad v_1 = \frac{-\sin nx}{n^2}$$

$$u_3 = -6 \quad \downarrow \quad v_2 = \frac{\cos nx}{n^3}$$

$$= \frac{1}{6\pi} \left[-(\pi^2 x - x^3) \frac{\cos nx}{n} + (\pi^2 - 3x^2) \frac{\sin nx}{n^2} - 6x \frac{\cos nx}{n} + 6 \frac{\sin nx}{n^3} \right]_0^{\pi}$$

$$b_n = -\frac{\cos n\pi}{n^3}$$

$$= -\frac{(-1)^n}{n^3}$$

$$\therefore f(x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx$$

$$\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \dots$$

③ $f(x) = x^2 + x$ in $(-\pi, \pi)$ (S) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = x + x^2$$

$$f(-x) = -x + (-x)^2$$

$$= -x + x^2$$

$$f(x) \neq f(-x)$$

$$f(x) \neq -f(x)$$

$\therefore f(x)$ is neither even nor odd.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x + x^2 dx$$

$$= \frac{1}{\pi} \left(\frac{x^2}{2} + \frac{x^3}{3} \right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right]$$

$$\begin{aligned}
 a_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x + x^2 \cos nx \, dx \\
 &= \frac{1}{\pi} \left[(x + x^2) \frac{\sin nx}{n} + (1 + 2x) \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[(1 + 2\pi) \frac{\cos \pi n}{n^2} - (1 + 2(-\pi)) \frac{\cos n(-\pi)}{n^2} \right] \\
 &= \frac{1}{\pi} \left[(1 + 2\pi) \frac{(-1)^n}{n^2} - (1 - 2\pi) \frac{(-1)^n}{n^2} \right] \\
 &= \frac{(-1)^n}{n^2 \pi} [x + 2\pi - y + 2\pi] \\
 &= \frac{(-1)^n 4\pi}{n^2 \pi} \\
 a_n &= \frac{4(-1)^n}{n^2} \\
 b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} x + x^2 \sin nx \, dx \\
 &= \frac{1}{\pi} \left[(x + x^2) \frac{\cos nx}{n} + (1 + 2x) \frac{\sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[-(\pi + \pi^2) \frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} + (-\pi + \pi^2) \frac{\cos n(-\pi)}{n} - \frac{2 \cos n(-\pi)}{n^3} \right]
 \end{aligned}$$

$$b_n = \frac{1}{\pi} \left[\frac{-\pi(-1)^n}{n} - \frac{\pi^2(-1)^n}{n} - \frac{\pi(-1)^n}{n} + \frac{\pi^2(-1)^n}{n} \right]$$

$$= \frac{1}{\pi} \left[\frac{-2\pi(-1)^n}{n} \right]$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ -\frac{2}{n} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi^2}{3 \cdot 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

If $x=0, x=\pi$

$$\pi + \pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$$

$$\pi + \pi^2 - \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$

If $x = \frac{\pi}{2}$ (discontinuous at end point)

$$\frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$

$$\frac{\pi + \pi^2 + \pi + \pi^2}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3 \cdot 4} = \frac{1}{12} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{1}{12} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Half range sine series in the interval $(0, \pi)$ $(0, l)$

Formula : $(0, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx dx$$

Formula: $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{b-a} \int_a^b f(x) \frac{\sin n\pi x}{l} dx$$

Problems:

① $f(x) = x(\pi-x)$ in $0 < x < \pi$.

Sol:

$$f(x) = x(\pi-x)$$

$$= x\pi - x^2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx dx$$

$$u = x\pi - x^2 \quad dv = \sin nx$$

$$u_1 = \pi - 2x \quad \downarrow \quad v = -\frac{\cos nx}{n}$$

$$u_2 = -2 \quad \downarrow \quad v_1 = -\frac{\sin nx}{n^2}$$

$$\downarrow \quad v_2 = \frac{\cos nx}{n^3}$$

$$= \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \sin nx dx$$

$$= \frac{2}{\pi} \left[(x\pi - x^2) \frac{\cos nx}{n} + (\pi - 2x) \frac{\sin nx}{n^2} \right. \\ \left. - \frac{2 \cos nx}{n^3} \right]_0^{\pi}$$

$$b_n = \frac{4}{\pi n^3} [1 - (-1)^n]$$

$$= \begin{cases} \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{(2n-1)^3}$$

$$\frac{\pi}{2} \left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{8\pi^3}{82} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

① Obtain the sine series for the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{l}{2} \\ l-x, & \frac{l}{2} \leq x \leq l \end{cases}$$

Sol:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin nx \, dx$$

$$\begin{array}{l}
 u = x \quad dv = \frac{\sin n\pi x}{l} \quad u = l-x \\
 u_1 = 1 \quad v = -\frac{\cos n\pi x}{\frac{n\pi}{l}} \quad u_1 = -1 \\
 u_2 = 0 \quad v_1 = \frac{-\sin n\pi x}{(\frac{n\pi}{l})^2} \quad u_2 = 0
 \end{array}$$

$$b_n = \frac{2}{l} \left[\left(-x \frac{\cos n\pi x}{\frac{n\pi}{l}} + \frac{\sin n\pi x}{(\frac{n\pi}{l})^2} \right) \Big|_0^{l/2} + \left(-(l-x) \frac{\cos n\pi x}{\frac{n\pi}{l}} - \frac{\sin n\pi x}{(\frac{n\pi}{l})^2} \right) \Big|_{l/2}^l \right]$$

$$b_n = \frac{2}{l} \left[\frac{-l}{2} \frac{1}{n\pi} + \left(\frac{l}{2} \right) \frac{1}{n\pi} \right]$$

$$b_n = \frac{2}{l} \left[-\frac{l^2}{2} \frac{\cos n\pi}{n\pi} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{l^2 \sin n\pi}{n^2\pi^2} + \frac{l^2}{2n^2\pi^2} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{2}{l} \left(\frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right)$$

$$b_n = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \frac{\sin n\pi x}{l}$$

$$\textcircled{3} f(x) = (2-x) \text{ in } (0, 2).$$

Sol:

$$b_n = \frac{2}{2} \int_0^2 (2-x) \sin \frac{n\pi x}{2} dx$$

$$u = 2-x \quad dv = \sin \frac{n\pi x}{2}$$

$$v = -\cos \frac{n\pi x}{2}$$

$$u_1 = -1$$

$$v_1 = -\frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$$

$$b_n = \left[-(2-x) \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= 2 \frac{2}{n\pi}$$

$$= \frac{4}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi x}{2}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2 \pi} [-1 + (-1)^n]$$

$$a_n = \begin{cases} \frac{-4}{n^2 \pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [1 - (-1)^n] \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{-4}{n^2 \pi} \cos nx$$

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2 \pi}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} - \frac{\pi^3}{3} \right]$$

Half range cosine series in the interval $(0, \pi), (0, l)$

Formula: $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx dx$$

Formula: $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

(1) Find the half range cosine series of $f(x) = x$ in $(0, \pi)$.

Sol:

$$f(x) = x.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2 \pi} [-1 + (-1)^n]$$

$$a_n = \begin{cases} \frac{-4}{n^2 \pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [1 - (-1)^n] \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{-4}{n^2 \pi} \cos nx$$

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2 \pi}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} - \frac{\pi^3}{3} \right]$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \cos nx \, dx$$

$$u = x\pi - x^2 \quad \int dv = \int \cos nx \, dx$$

$$u_1 = \pi - 2x \quad v = \frac{\sin nx}{n}$$

$$u_2 = -2 \quad v_1 = -\frac{\cos nx}{n^2}$$

$$u_3 = 0 \quad v_2 = -\frac{\sin nx}{n^3}$$

$$a_n = \frac{2}{\pi} \left[(x\pi - x^2) \frac{\sin nx}{n} + (\pi - 2x) \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi \frac{(-1)^n}{n^2} - \frac{\pi}{n^2} \right]$$

$$= \frac{2\pi}{\pi n^2} \left[-(-1)^n - 1 \right]$$

$$= \frac{-2}{n^2} \left[1 + (-1)^n \right]$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{n^2} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=2,4,6,\dots}^{\infty} \frac{-4}{n^2} \cos nx$$

③ Find the half range cosine series of

$$f(x) = (x-1)^2, \quad 0 < x < 1.$$

$$a_0 = \frac{2}{1} \int_0^1 f(x) dx$$

$$= 2 \int_0^1 (x-1)^2 dx$$

$$= 2 \left[\frac{(x-1)^3}{3} \right]_0^1$$

$$= \frac{2}{3} \left[0 + \frac{1}{3} \right]$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{1} \int_0^1 f(x) \cos \frac{n\pi x}{1} dx$$

$$= 2 \int_0^1 (x-1)^2 \cos \frac{n\pi x}{1} dx$$

$$u = (x-1)^2 \quad \left| \begin{array}{l} du = 2(x-1) dx \\ dv = \int \cos \frac{n\pi x}{1} dx \end{array} \right.$$

$$u_1 = 2(x-1) \quad v = \frac{\sin n\pi x}{n\pi}$$

$$u_2 = 2 \quad v_1 = -\frac{\cos n\pi x}{n^2 \pi^2}$$

$$v_2 = -\frac{\sin n\pi x}{n^3 \pi^3}$$

$$a_n = 2 \left[(x-1)^2 \frac{\sin n\pi x}{n\pi} + 2(x-1) \frac{\cos n\pi x}{n^2 \pi^2} - 2 \frac{\sin n\pi x}{n^3 \pi^3} \right]$$

$$= 2 \left[\frac{2}{n^2 \pi^2} \right]$$

$$= \frac{4}{n^2 \pi^2}$$

Formula: Complex form $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{where } c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} dx$$

Formula: complex form $(-l, l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{l}}$$

$$\text{where } c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{i n \pi x}{l}} dx$$

① find the complex form $f(x) = e^{ax}$, $-\pi < x < \pi$

$$\text{in the form } e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a + in}{a^2 + n^2} e^{inx}$$

$$\& \text{ hence prove that } \frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2}$$

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$\begin{aligned}
 C_n &= \frac{1}{2\pi} \left[\frac{e^{(a-in)\pi}}{a-in} - \frac{e^{(a-in)(-\pi)}}{a-in} \right] \\
 &= \frac{1}{2\pi(a-in)} \left[e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi} \right] \\
 &= \frac{1}{2\pi(a-in)} \left[e^{a\pi} (\cos n\pi - i \sin n\pi) - e^{-a\pi} (\cos n\pi + i \sin n\pi) \right] \\
 &= \frac{1}{2\pi(a-in)} \left[e^{a\pi} (-1)^n - e^{-a\pi} (-1)^n \right] \\
 &= \frac{(-1)^n}{2\pi(a-in)} (e^{a\pi} - e^{-a\pi})
 \end{aligned}$$

$$= \frac{(-1)^n}{\pi(a-in)} \frac{\sinh a\pi}{a+in}$$

$$C_n = \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

Put $x=0$.

$$1 = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)}$$

Equating the real parts, we get

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{a^2 + n^2}$$

$$\therefore \frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2}$$

2. $f(x) = e^{ax}$ in $(-l, l)$

Sol: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^l e^{ax} e^{-\frac{in\pi x}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^l e^{\left(a - \frac{in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l} \left[\frac{e^{\left(a - \frac{in\pi}{l}\right)x}}{a - \frac{in\pi}{l}} \right]_{-l}^l$$

$$= \frac{1}{2l} \left[\frac{e^{\frac{al - in\pi}{l}} \cdot l}{a - \frac{in\pi}{l}} - \frac{e^{\frac{al - in\pi}{l} \cdot (-l)}}{a - \frac{in\pi}{l}} \right]$$

$$= \frac{1}{2} \left[\frac{e^{al - in\pi} - e^{-al + in\pi}}{a - \frac{in\pi}{l}} \right]$$

$$C_n = \frac{1}{2l(al - in\pi)} \left[e^{al} \cdot (-1)^n - e^{-al} \cdot (-1)^n \right]$$

$$= \frac{(-1)^n}{2l(al - in\pi)} \left[e^{al} - e^{-al} \right]$$

$$C_n = \frac{(-1)^n \sinh al}{l(al - in\pi)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh al}{(al - in\pi)} e^{+ \frac{in\pi x}{l}}$$

③ Find the complex form of the fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$.

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$2l=2 \\ l=1.$$

$$C_n = \frac{1}{b-a} \int_a^b f(x) e^{-\frac{in\pi x}{l}} dx$$

$$C_n = \frac{1}{b-a} \int_a^b f(x) e^{-\frac{in\pi x}{l}} dx$$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-x} \cdot e^{-\frac{in\pi x}{1}} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{-(x + \frac{in\pi}{1})x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^1$$