

UNIT III – APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially displaced in the position $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ and then released from rest. Find the displacement y at any distance x from one end at time t .

Sol:



The displacement $y(x,t)$ is from $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The Conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

(iv) $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

$\sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$

The suitable sol. is

$y(x,t) = (C_1 \cos px + C_2 \sin px) (C_3 \cos pat + C_4 \sin pat)$

Apply (i) $y(0,t) = 0$

$C_1 (C_3 \cos pat + C_4 \sin pat) = 0$

$C_1 = 0$

Apply (ii) in (i) $x=l$

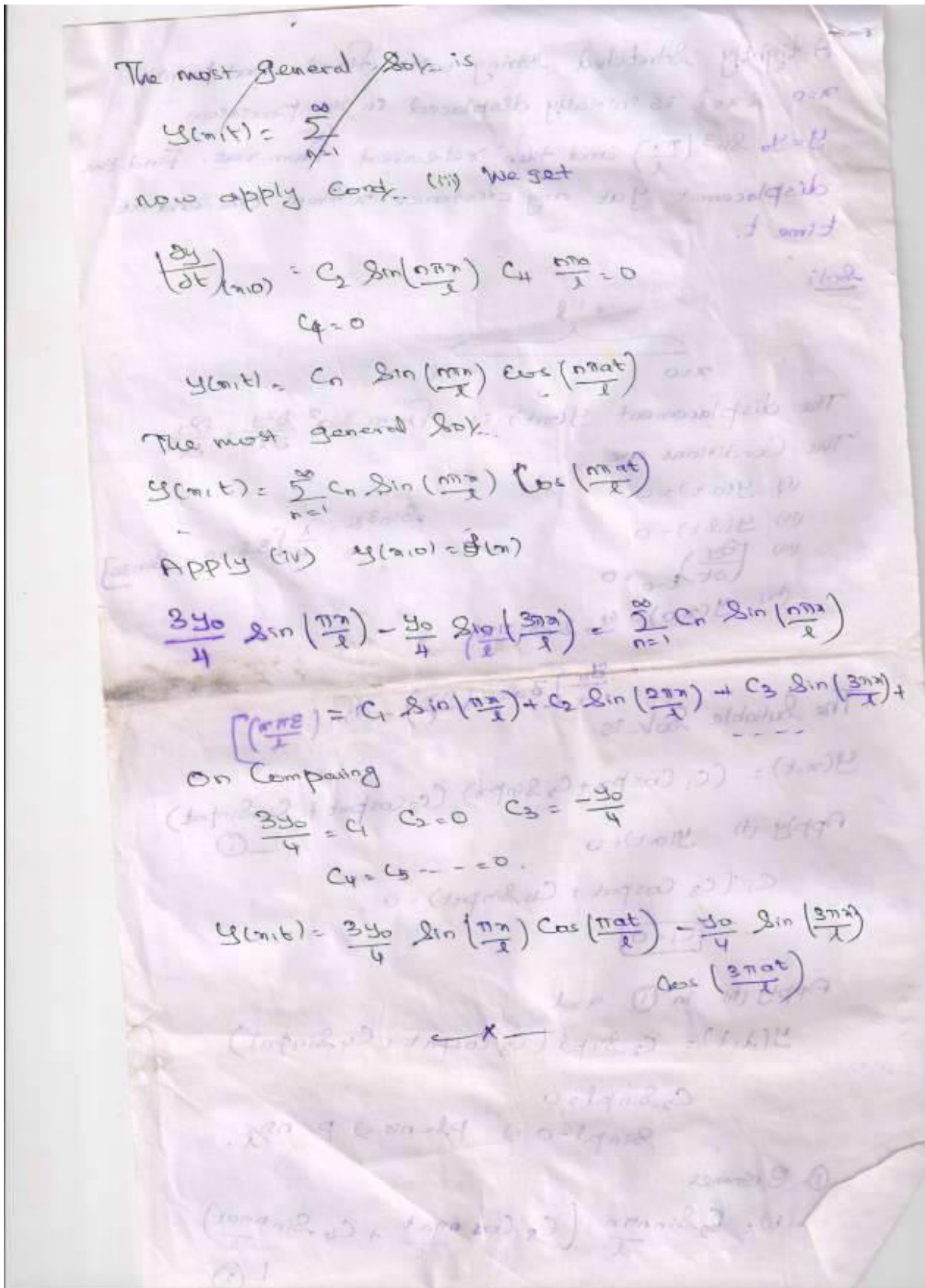
$y(l,t) = C_2 \sin pl (C_3 \cos pat + C_4 \sin pat)$

$C_2 \sin pl = 0$

$\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$

① becomes

$y(x,t) = B_2 \sin \frac{n\pi x}{l} \left(C_3 \cos \frac{n\pi a t}{l} + C_4 \sin \frac{n\pi a t}{l} \right)$



A string of length $2l$ is fastened at both ends. The mid pt. of the string is taken to a height b and then released from rest in that position. Show that the displacement is $-x$

A tightly stretched string of length l has its ends fastened at $x=0$ and $x=l$. The mid point of the string is then taken to a height h and then released from rest in that position. Obtain an expression for the displacement of the string at any subsequent time.

Sol: The eqn. to be solved is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

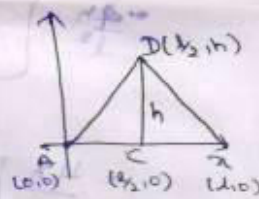
From the given problem, we get the following boundary and initial conditions.

(1) $y(0,t) = 0$ (2) $y(l,t) = 0$ (3) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$

Eqn. of AD is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x-0}{\frac{l}{2}-0} = \frac{y-0}{h-0}$$

$$2hx = \frac{l}{2} y \Rightarrow y = \frac{4hx}{l}$$



Eqn. of DB is

$$\frac{x-\frac{l}{2}}{l-\frac{l}{2}} = \frac{y-h}{0-h} \Rightarrow \frac{x-\frac{l}{2}}{\frac{l}{2}} = \frac{y-h}{-h}$$

$$y-h = -\frac{2h}{l} \left(x-\frac{l}{2}\right) \Rightarrow y = h - \frac{2hx}{l} + h$$

$$y = 2h \left(1 - \frac{x}{l}\right) = \frac{2h}{l} (l-x)$$

$$y(x,0) = \begin{cases} \frac{4hx}{l} & 0 < x < \frac{l}{2} \\ \frac{2h}{l} (l-x) & \frac{l}{2} < x < l \end{cases}$$

$$15) y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

Applying cond. (ii) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = \begin{cases} \frac{2hx}{l} & 0 < x < \frac{l}{2} \\ \frac{2h(1-x)}{l} & \frac{l}{2} < x < l \end{cases}$$

To find C_n : Expand the value in a half-range sine series

$$\left. \begin{aligned} \frac{2hx}{l}, & 0 < x < \frac{l}{2} \\ \frac{2h}{l}(1-x), & \frac{l}{2} < x < l \end{aligned} \right\} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$C_n = \frac{2}{l} \left[\int_0^{\frac{l}{2}} \frac{2hx}{l} \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l \frac{2h(1-x)}{l} \sin \frac{n\pi x}{l} dx \right]$$

$$C_n = \frac{4h}{l^2} \left[\int_0^{\frac{l}{2}} x \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l (1-x) \sin \frac{n\pi x}{l} dx \right]$$

$$= \frac{4h}{l^2} \left[-x \frac{l}{n\pi} \cos \frac{n\pi x}{l} + \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l} \right]_0^{\frac{l}{2}}$$

$$+ \frac{4h}{l^2} \left[-(1-x) \frac{l}{n\pi} \cos \frac{n\pi x}{l} - \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi x}{l} \right]_{\frac{l}{2}}^l$$

$$= \frac{4h}{l^2} \left[\left(-\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right) - (0+0) \right]$$

$$+ \frac{4h}{l^2} \left[(-0-0) - \left(-\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right) \right]$$

$$= \frac{4h}{l^2} \left[\frac{-l^2}{2n\pi} \cos \frac{n\pi}{2} + \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi}{2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \left(\frac{l}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right]$$

$$= \frac{8h}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

-X-

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin \left(\frac{(2n-1)\pi x}{2l} \right) \cos \left(\frac{(2n-1)\pi ct}{2l} \right)$$

Sol: The Wave eq_n is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

- ① $y(0,t) = 0$
- ② $y(2l,t) = 0$
- ③ $\left(\frac{\partial y}{\partial t} \right)_{x=0} = 0$

Eq_n of AB

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{b-0} = \frac{x-0}{l-0} \Rightarrow y = \frac{b}{l}x$$

Eq of BA

$$\frac{y-b}{0-b} = \frac{x-l}{2l-l} \Rightarrow y = b \left(\frac{2l-x}{l} \right)$$

$$y(x,0) = \begin{cases} \frac{bx}{l} \\ b \left(\frac{2l-x}{l} \right) \end{cases}$$

$$y(x,t) = (C_1 \cos pt + C_2 \sin pt) (C_3 \cos \frac{pct}{l} + C_4 \sin \frac{pct}{l})$$

$y(0,t) = 0 \Rightarrow C_1 = 0$

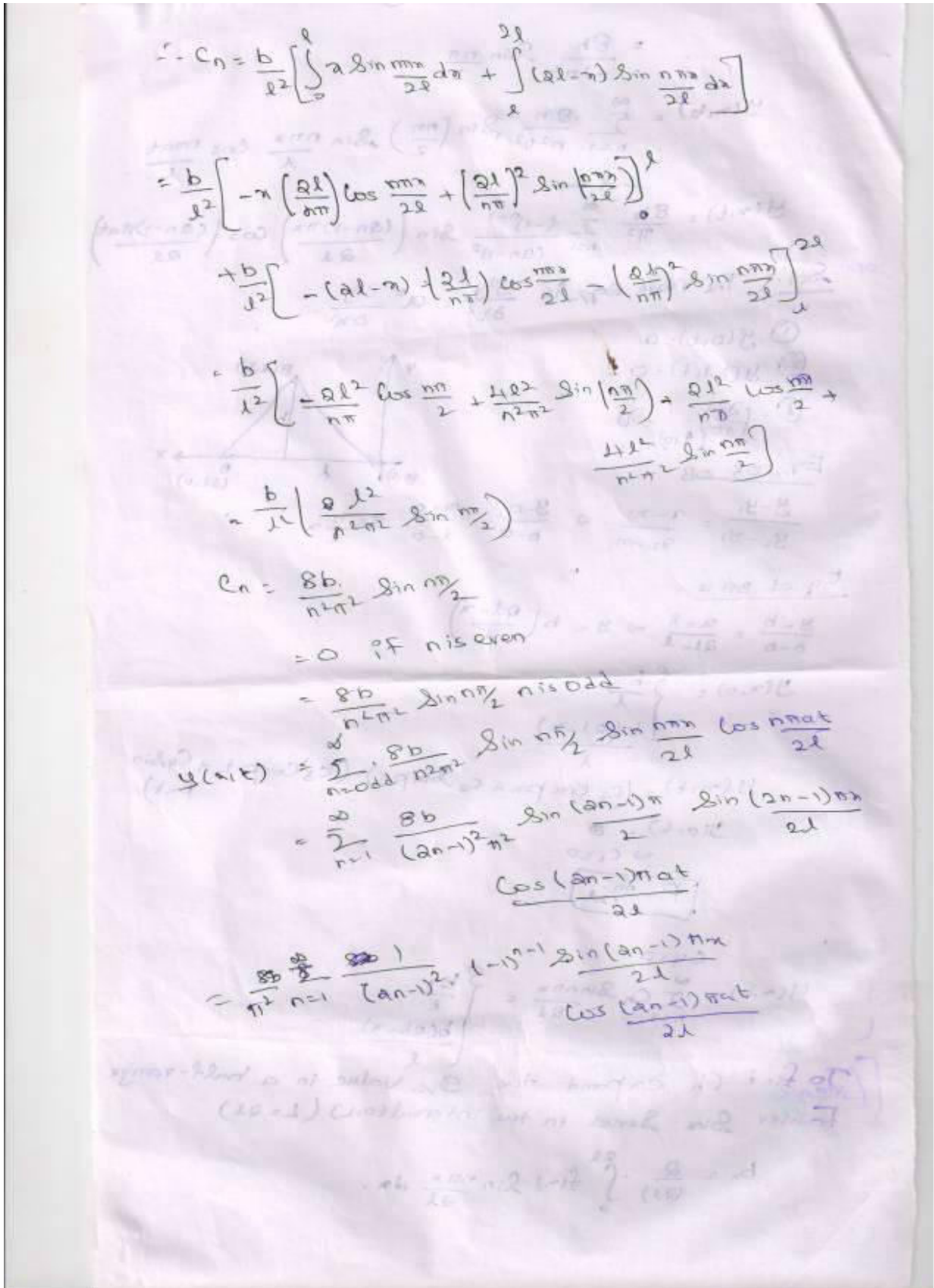
$P = \frac{n\pi}{2l}$

$C_4 = 0$

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{2l} = \begin{cases} \frac{bx}{l} \\ b \left(\frac{2l-x}{l} \right) \end{cases}$$

To find C_n expand the R.H. value in a half-range Fourier sine series in the interval $(0, 2l)$ ($L=2l$)

$$b_n = \frac{2}{2l} \int_0^{2l} f(x) \sin \frac{n\pi x}{2l} dx.$$



Problems on Vibrating String with non-zero initial velocity.

The boundary and initial conditions of the deflection $y(x,t)$ are

① $y(0,t) = 0$ ② $y(l,t) = 0$ ③ $y(x,0) = 0$ ④ $\frac{\partial y}{\partial t}(x,0) = f(x)$

The suitable solⁿ is

$$y(x,t) = (C_1 \cos px + C_2 \sin px) (C_3 \cos pat + C_4 \sin pat)$$

Apply condⁿ ① we set $C_1 = 0$ - ①

② $p = \frac{n\pi}{l}$

③ $C_3 = 0$

The most general solⁿ is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$$

Apply ④ conⁿ.

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x)$$

where $B_n = C_n \frac{n\pi a}{l}$

$$B_n = b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$\therefore C_n = \frac{l}{n\pi a} B_n$$

A tightly stretched string with fixed end pts $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$ show that the displacement is

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi at}{l}$$

Sol 11 The wave eqn. is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Boundary & initial Cond.

- ① $y(0,t) = 0 \quad \forall t$
- ② $y(l,t) = 0$ ③ $y(x,0) = 0 \quad 0 < x < l$
- ④ $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l-x) \quad 0 < x < l.$

The suitable sol. is

$$y(x,t) = (C_1 \cos px + C_2 \sin px) \cdot (C_3 \cos pat + C_4 \sin pat) \quad \text{--- (I)}$$

Apply Cond. ① in I

$$y(0,t) = C_1 [C_3 \cos pat + C_4 \sin pat] = 0$$

$$C_3 \cos pat + C_4 \sin pat \neq 0$$

$$\therefore \boxed{C_1 = 0}$$

Substitute $C_1 = 0$ in I

$$y(x,t) = C_2 \sin px (C_3 \cos pat + C_4 \sin pat) \quad \text{--- (II)}$$

Apply Cond. ② in II

$$y(l,t) = C_2 \sin pl (C_3 \cos pat + C_4 \sin pat) = 0$$

$$\text{Here } C_3 \cos pat + C_4 \sin pat \neq 0.$$

$$C_2 \sin pl = 0 \Rightarrow C_2 = 0 \text{ (or) } \sin pl = 0$$

Suppose $C_2 = 0$ already we have $C_1 = 0$.

$$\therefore C_2 \neq 0. \quad \sin pl = 0 = \sin n\pi$$

$$pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

Now, sub $p = \frac{n\pi}{l}$ in eqn. II

$$y(x,t) = C_2 \sin \frac{n\pi x}{l} \left(C_3 \cos \frac{n\pi at}{l} + C_4 \sin \frac{n\pi at}{l} \right) \quad \text{--- (III)}$$

Apply Cond. ③ in III

$$y(x,0) = C_2 \sin \frac{n\pi x}{l} \quad C_3 = 0.$$

$$C_2 C_3 \sin \frac{n\pi x}{l} = 0$$

$$\sin \frac{n\pi x}{l} \neq 0 \text{ (it is defined for all } n)$$

$$C_2 \neq 0.$$

$$\therefore C_3 = 0.$$

Sub $C_3 = 0$ in eq B

$$y(x,t) = C_2 C_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}$$

$$= C_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l} \quad \text{--- (2)}$$

The most general sol.:

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l} \quad \text{--- (2)}$$

Before apply condy. ~~if~~ diff (2)

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \left(\frac{n\pi a}{l}\right) \cos \frac{n\pi a t}{l}$$

Apply (4) condy.

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \left(\frac{n\pi a}{l}\right) = \lambda x(1-x)$$

$$= \sum_{n=1}^{\infty} \left(C_n \frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$B_n = C_n \frac{n\pi a}{l}$$

To find B_n ! Expand $\lambda x(1-x)$ in half range sine series

$$\lambda x(1-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

$$B_n = \frac{2}{l} \int_0^l \lambda x(1-x) \sin \frac{n\pi x}{l} dx.$$

$$= \frac{2\lambda}{l} \left[- (2n-2)^2 \left(\frac{l}{n\pi}\right) \cos \frac{n\pi x}{l} + (2-2x) \left(\frac{l}{n\pi}\right)^2 \sin \frac{n\pi x}{l} - 2 \left(\frac{l}{n\pi}\right)^3 \cos \left(\frac{n\pi x}{l}\right) \right]$$

$$= \frac{2\lambda}{l} \left[\left(-2 \left(\frac{l}{n\pi}\right)^3 (-1)^n\right) - 1 - 2 \left(\frac{l}{n\pi}\right)^3 \right]$$

$$= \frac{2\lambda}{l} \left[-2 \left(\frac{l}{n\pi}\right)^3 (-1)^n + 2 \left(\frac{l}{n\pi}\right)^3 \right]$$

$$= \frac{8\lambda}{l} \times 2 \left(\frac{l}{n\pi}\right)^3 [1 - (-1)^n]$$

$$= \frac{4\lambda l^2}{n^3 \pi^3} (1 - (-1)^n) = C_n \left(\frac{n\pi a}{l}\right)$$

$$C_n = \left(\frac{l}{n\pi a}\right) \left(\frac{4\lambda l^2}{n^3 \pi^3}\right) (1 - (-1)^n)$$

$$C_n = \begin{cases} 0 & n = \text{even} \\ \frac{8\lambda l^3}{a n^4 \pi^4} & n = \text{odd} \end{cases}$$

$$y(x,t) = \frac{8\lambda l^3}{a n^4} \sum_{n=\text{odd}} \frac{1}{n^4} \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}$$

$$= \frac{8\lambda l^3}{a n^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi a t}{l}$$

- X -

Q) A tightly stretched string of length l is initially at rest in its equilibrium position and each of its pts is given the velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find the displacement

Sol: The wave eq. is $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The cond. are (1) $y(0,t) = 0$ (2) $y(l,t) = 0$

(3) $y(x,0) = 0$ (4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$

$$\frac{3V_0}{4} \sin\left(\frac{\pi x}{l}\right) - \frac{V_0}{4} \sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right) + \dots$$

$$B_1 = \frac{3V_0}{4} \quad B_3 = -\frac{V_0}{4}$$

$$B_1 = \frac{3V_0 l}{4\pi a} \quad B_3 = -\frac{V_0 l}{4\pi a}$$

$$B_3 = C_3 \sin\left(\frac{3\pi a}{l}\right) = -\frac{V_0}{4}$$

$$C_3 = -\frac{4V_0}{12\pi a}$$

$$y(x,t) = \frac{3V_0 l}{4\pi a} \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi a t}{l}\right) - \frac{V_0 l}{12\pi a} \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi a t}{l}\right)$$

⑧ A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its pts a velocity. Given by

$$v = \begin{cases} cx & \text{if } 0 \leq x \leq \frac{l}{2} \\ c(l-x) & \frac{l}{2} \leq x \leq l \end{cases}$$

Find the displacement

sol: The wave eqn.

boundary cond.

$$y(0,t) = 0 \quad y(l,t) = 0 \quad y(x,0) = 0$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} cx & 0 \leq x \leq \frac{l}{2} \\ c(l-x) & \frac{l}{2} \leq x \leq l \end{cases}$$

$$y(x,t) = (C_1 \cos pn + C_2 \sin pn) (C_3 \cos p a t + C_4 \sin p a t)$$

One dimensional Heat eqn.

The One dimensional heat eqn is

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

$$\alpha^2 = \frac{k}{\rho c} \rightarrow \text{Diffusivity of the material}$$

$k \rightarrow$ Thermal Conductivity

$\rho \rightarrow$ Density

$c \rightarrow$ specific heat.

32 → 218
Sec-B 22, 50 418

Mech-D Anal, 18, 100

Temperature fm. is $u(x,t)$

— x —

Note: $\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$$A = \alpha^2 \quad B = 0 \quad C = 0$$

$$B^2 - 4AC = 0 = 0 - 4\alpha^2(0) = 0$$

The One dimensional heat eqn is parabolic

Write all possible sol. of one dimensional heat eqn.

$$\text{(1) } u(x,t) = [Ae^{px} + Be^{-px}] Ce^{\alpha^2 pt}$$

$$\text{(2) } u(x,t) = [A \cos px + B \sin px] Ce^{-\alpha^2 pt}$$

$$\text{(3) } u(x,t) = (Ax + B)C$$

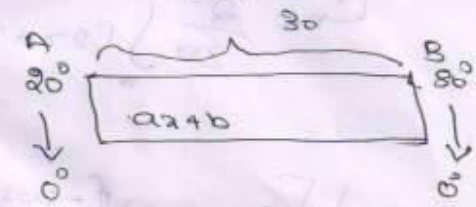
A rod 30 cm long has its ends A and B kept at 20°C and 80°C respectively. Unst)

Steady State Conditions prevail. The temp at each end is then suddenly reduced to 0°C and kept so on. Find the resulting

CIVIL

Sol: The temp. $u(x,t)$ is from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$



The condns are

(i) $u(0,t) = 0$

(ii) $u(30,t) = 0$

(iii) $u(x,0) = a_2 + b = 2x + 20$

$$a = \frac{80 - 20}{30} = \frac{60}{30} = 2$$

$$b = \frac{0 \times 0}{30} = 0$$

The soln is

$$u(x,t) = [A \cos px + B \sin px] C e^{-p^2 \alpha^2 t}$$

Apply (i) Put $x=0$

$$0 = A C e^{-p^2 \alpha^2 t}$$

$$A = 0$$

Apply (ii) Put $x=30$

$$0 = B \sin(30p) C e^{-p^2 \alpha^2 t}$$

$$B \sin(30p) = 0$$

$$\sin(30p) = 0 \Rightarrow 30p = n\pi \Rightarrow p = \frac{n\pi}{30}$$

\therefore (i) becomes

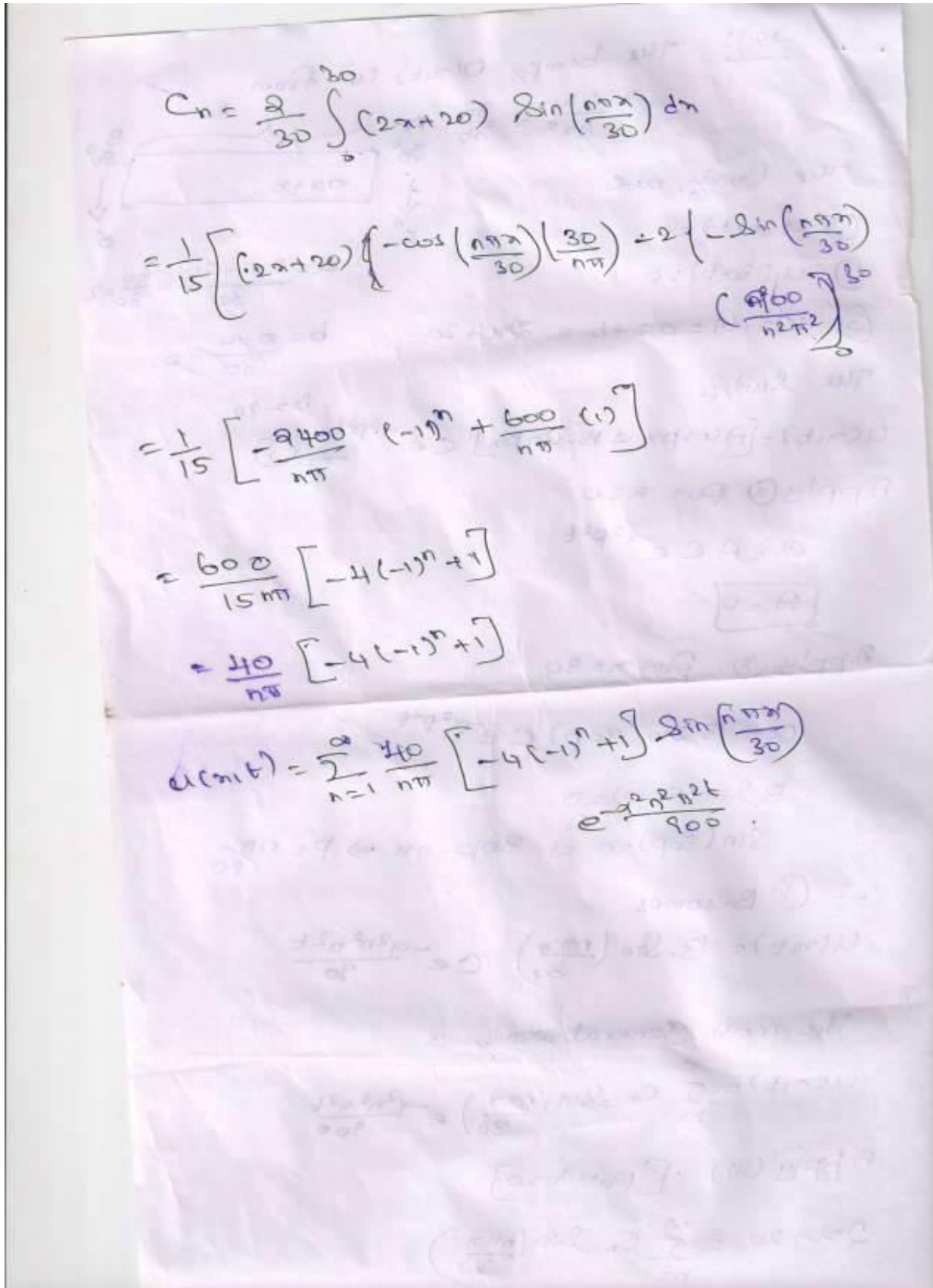
$$u(x,t) = B \sin\left(\frac{n\pi x}{30}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

The most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

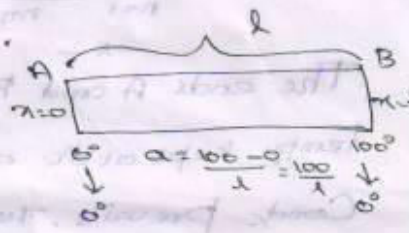
Apply (iii) [Put $t=0$]

$$2x + 20 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{30}\right)$$



A rod of length l has its ends A and B kept at 0°C and 100°C resp. Until steady state conditions prevail. If the temp. at B is reduced suddenly to 0°C and kept so while that of A is maintained. Find the temperature $u(x,t)$.

Solⁿ



The temp. $u(x,t)$ is from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

The cond. are

- ① $u(0,t) = 0$ ② $u(l,t) = 0$ ③ $u(x,0) = \alpha x + b = \frac{100}{l}x$

The suitable solⁿ is

$$u(x,t) = (A \cos px + B \sin px) C e^{-\alpha^2 p^2 t}$$

Apply ① $x=0$

$$A(C e^{-\alpha^2 p^2 t}) = 0 \Rightarrow A = 0$$

Apply ②

$$u(l,t) = A \cos pl + B \sin pl (C e^{-\alpha^2 p^2 t}) = 0$$

$$\sin pl = 0 = \sin n\pi \Rightarrow pl = n\pi \quad \boxed{p = \frac{n\pi}{l}}$$

$$u(x,t) = \frac{B \sin n\pi x}{l} C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most gen^r. $u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$ ④

Apply ③ in ④

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$C_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{200}{l^2} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{200}{l^2} \left[x \left(-\cos\left(\frac{n\pi x}{l}\right) \right) \left(\frac{l}{n\pi} \right) + \frac{l^2}{n^2 \pi^2} \sin\left(\frac{n\pi x}{l}\right) \right]_0^l$$

$$= \frac{200}{l^2} \left[-\frac{l^2}{n\pi} (-1)^n \right] = -\frac{200(-1)^n}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

The ends A and B of a rod 40cm long have their temp. kept at 0°C and 80°C resp. Until steady state Cond. prevails, the temp. of the end B is then suddenly reduced to 40°C and kept so. while that of the end A is kept at 0°C find the subsequent temp. distribution $u(x,t)$ in the rod.

$a = \frac{80-0}{40} = 2$
 $b = 0$
 $a+b = 2x = 40$

The temp. $u(x,t)$ is from $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

The Cond. are:

- ① $u(0,t) = 0$
- ② $u(40,t) = 40$
- ③ $u(x,0) = a x + b = 2x$

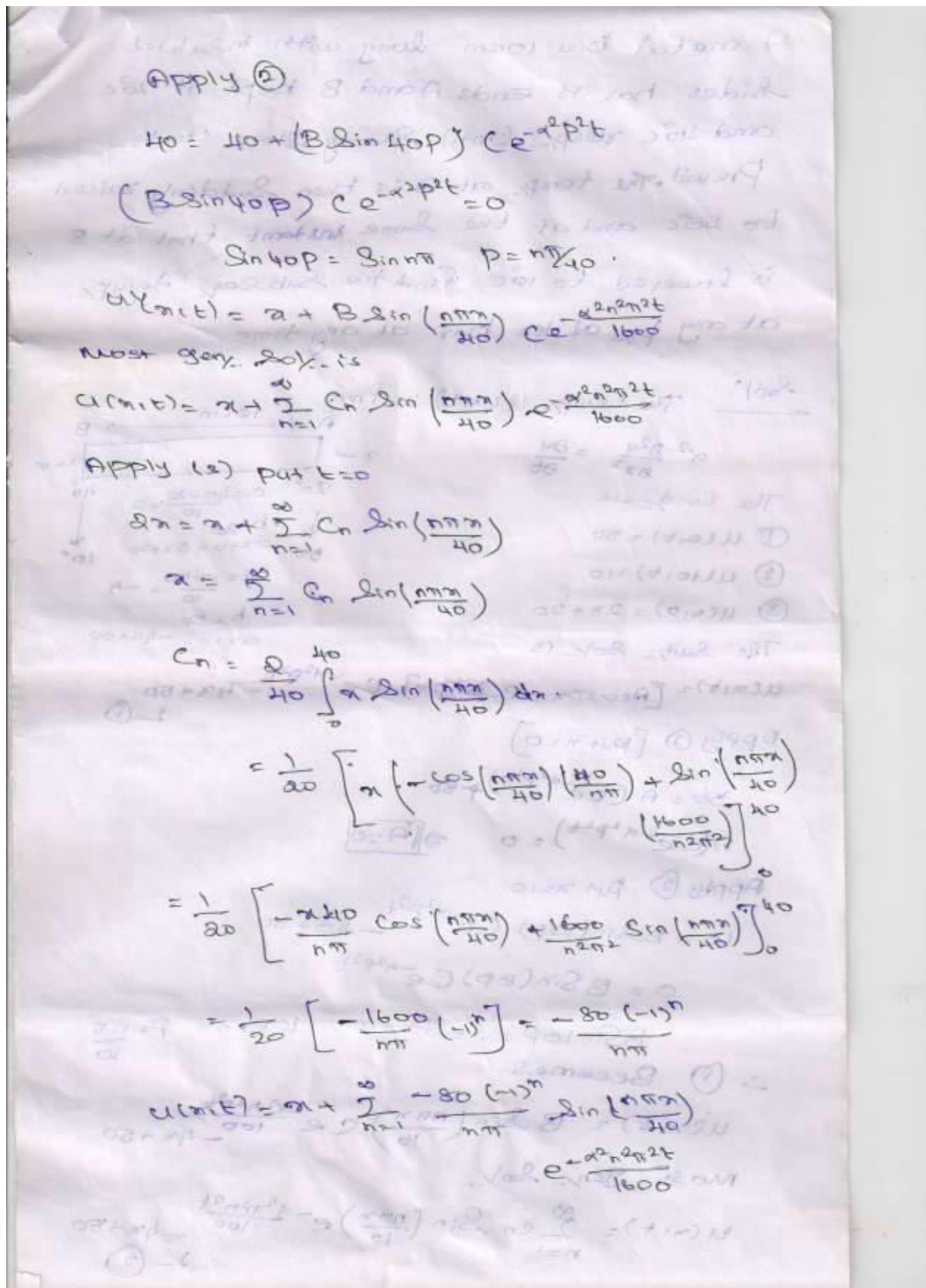
The suitable sol.

$$u(x,t) = (A \cos p x + B \sin p x) e^{-\alpha^2 p^2 t} + U_s(x)$$

$$U_s(x) = a x + b = 2x$$

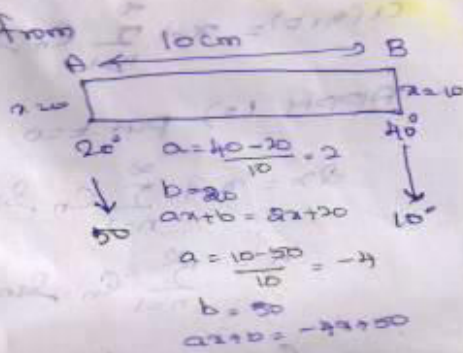
$$u(x,t) = 2x + (A \cos p x + B \sin p x) e^{-\alpha^2 p^2 t} \quad \text{--- (1)}$$

Apply ①



A metal bar 10cm long with its sides has its ends A and B kept at 20°C and 40°C resp. Until steady state cond. prevail. The temp. at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subseq. temp. at any pt. of the bar at any time.

Sol: The temp. $u(x,t)$ is from



$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

The cond. are

- ① $u(x,t) = 50$
- ② $u(10,t) = 10$
- ③ $u(x,0) = 2x + 20$

The suit. sol. is

$$u(x,t) = [A \cos px + B \sin px] C e^{-\alpha^2 p^2 t} - 4x + 50 \quad \text{--- (1)}$$

Apply ③ [put $x=0$]

$$50 = A (C e^{-\alpha^2 p^2 t} + 50)$$

$$A (C e^{-\alpha^2 p^2 t}) = 0 \Rightarrow \boxed{A=0}$$

Apply ② At $x=10$

$$10 = B \sin(10p) C e^{-\alpha^2 p^2 t} - 40 + 50$$

$$0 = B \sin(10p) C e^{-\alpha^2 p^2 t}$$

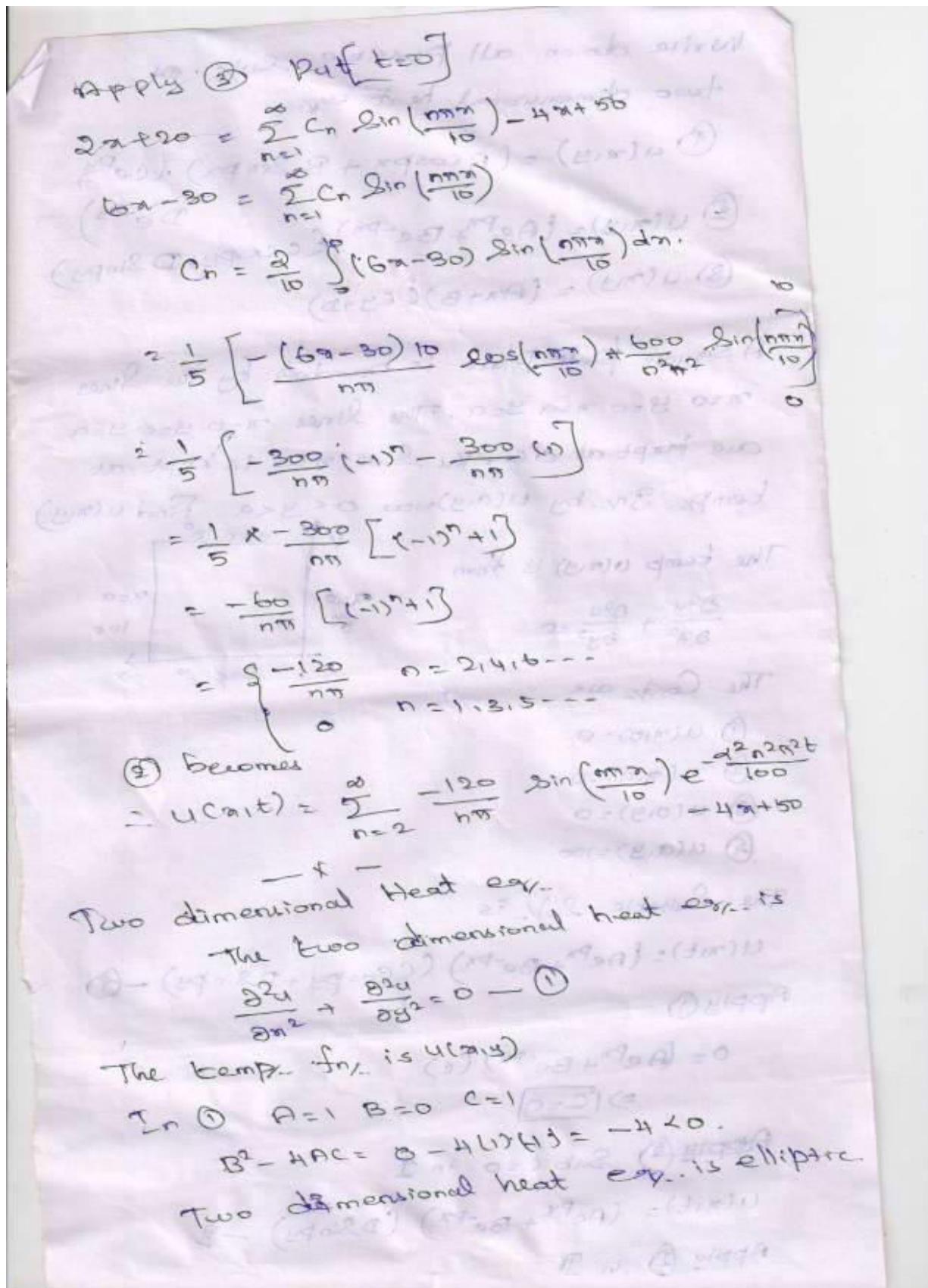
$$B \sin 10p = 0 = \sin n\pi \quad | \quad 10p = n\pi \quad p = \frac{n\pi}{10}$$

∴ ① Becomes.

$$u(x,t) = B \sin\left(\frac{n\pi x}{10}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} - 4x + 50$$

Most gen. sol.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} - 4x + 50 \quad \text{--- (2)}$$



Write down all possible sol_n of two dimensional heat eqn.

$$\textcircled{1} u(x,y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py})$$

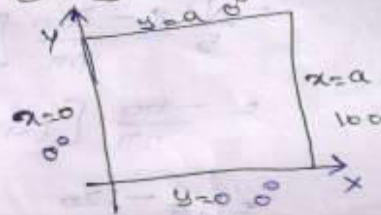
$$\textcircled{2} u(x,y) = (A e^{px} + B e^{-px}) (C \cos py + D \sin py)$$

$$\textcircled{3} u(x,y) = (Ax + B)(Cy + D)$$

A Square plate is bounded by the lines $x=0$, $y=0$, $x=a$, $y=a$. The lines $x=0$, $y=0$, $x=a$ are kept at 0°C . The side $x=a$ is kept at temp. 30°C by $u(a,y)=100$, $0 < y < a$. Find $u(x,y)$.

The temp $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



The Cond_n are

$$\textcircled{1} u(x,0) = 0$$

$$\textcircled{2} u(x,a) = 0$$

$$\textcircled{3} u(0,y) = 0$$

$$\textcircled{4} u(a,y) = 100$$

The suitable sol_n is

$$u(x,y) = (A e^{px} + B e^{-px}) (C \cos py + D \sin py) \quad \textcircled{1}$$

Apply $\textcircled{1}$

$$0 = (A e^{px} + B e^{-px}) (C)$$

$$\Rightarrow \boxed{C=0}$$

Apply $\textcircled{2}$ Sub $C=0$ in $\textcircled{1}$

$$u(x,y) = (A e^{px} + B e^{-px}) (D \sin py) \quad \textcircled{II}$$

Apply $\textcircled{3}$ in \textcircled{II}

$$D \sin pa = 0$$

$$\sin pa = 0 = \sin n\pi$$

$$p = \frac{n\pi}{a}$$

Sub $p = \frac{n\pi}{a}$ in (i)

$$u(x, y) = (A e^{\frac{n\pi y}{a}} + B e^{-\frac{n\pi y}{a}}) (D \sin \frac{n\pi x}{a}) \quad \text{--- (iii)}$$

APPLY (ii)

$$u(0, y) = (A+B) (D \sin \frac{n\pi x}{a}) = 0$$

$$B = -A$$

\therefore (iii) becomes

$$u(x, y) = A (e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}) D \sin \left(\frac{n\pi x}{a} \right)$$

The most gen. sol. is

$$u(x, y) = \sum_{n=1}^{\infty} D_n (e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}) \sin \left(\frac{n\pi x}{a} \right)$$

APPLY (iv) in (v)

$$u(a, y) = \sum_{n=1}^{\infty} D_n (e^{n\pi} - e^{-n\pi}) \sin \left(\frac{n\pi y}{a} \right) = 100$$

$$D_n (e^{n\pi} - e^{-n\pi}) = \frac{200}{a} \int_0^a 100 \sin \left(\frac{n\pi y}{a} \right) dy$$

$$= \frac{200}{a} \int_0^a \sin \left(\frac{n\pi y}{a} \right) dy$$

$$= \frac{200}{a} \left[-\cos \left(\frac{n\pi y}{a} \right) \left(\frac{a}{n\pi} \right) \right]_0^a$$

$$= \frac{200}{a} \left[+\frac{a}{n\pi} (-1)^n + \frac{a}{n\pi} \right]$$

$$= \frac{200}{a} \times \frac{a}{n\pi} [-(-1)^n + 1]$$

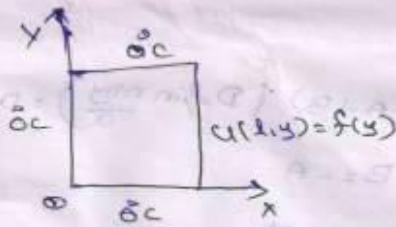
$$D_n [e^{n\pi} - e^{-n\pi}] = \begin{cases} 0 & n=2, 4, 6, \dots \\ \frac{400}{n\pi} & n=1, 3, 5, \dots \end{cases}$$

$$D_n = \frac{400}{n\pi [e^{n\pi} - e^{-n\pi}]} \quad n = 1, 3, 5, \dots$$

$$u(x,y) = \sum_{n=1,3,5} \frac{400}{n\pi [e^{n\pi} - e^{-n\pi}]} \left(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}} \right) \sin\left(\frac{n\pi y}{a}\right)$$

Finite Plate with Value Given in y-direction.

- ① $u(x,0) = 0$
- ② $u(x,l) = 0$
- ③ $u(0,y) = 0$
- ④ $u(l,y) = f(y)$

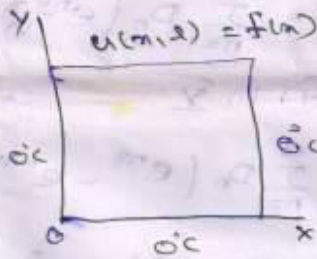


The Suitable Sol.

$$u(x,y) = (Ae^{px} + Be^{-px}) (C \cos py + D \sin py)$$

x-Direction

- ① $u(0,y) = 0$
- ② $u(l,y) = 0$
- ③ $u(x,0) = 0$
- ④ $u(x,l) = f(x)$



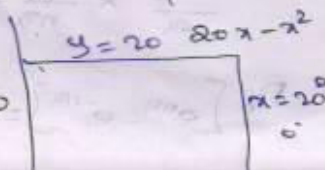
The Suitable Sol.

$$u(x,y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py})$$

A square plate is bounded by the lines $x=0, y=0, x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is $3x$.

by $u(x,20) = 3(20-x) = 20x - x^2$ $0 < x < 20$ while the other edges are kept at 0° . Find the steady state temp. dist. in the plate.

Sol!



The temp. $u(x,y)$ is from $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The cond. are
 (1) $u(0,y) = 0$ (2) $u(20,y) = 0$ (3) $u(x,0) = 0$ (4) $u(x,20) = 20x - x^2$

The suitable sol. is
 $u(x,y) = [A \cos px + B \sin px] [C e^{py} + D e^{-py}]$

Apply (1) $0 = A(C e^{py} + D e^{-py}) \Rightarrow A = 0$

Apply (2) $0 = B \sin(20p) (C e^{py} + D e^{-py})$
 $B \sin(20p) = 0 \Rightarrow 20p = n\pi \Rightarrow p = \frac{n\pi}{20}$

Apply (3)
 $0 = B \sin px (C + D)$
 $C + D = 0 \Rightarrow D = -C$
 \therefore (1) becomes
 $u(x,y) = B \sin px \cdot C [e^{py} - e^{-py}]$
 $= B \sin\left(\frac{n\pi x}{20}\right) C \left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}} \right]$

Most gen. sol.
 $u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right) \left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}} \right]$

Apply (4)
 $20x - x^2 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right) [e^{n\pi} - e^{-n\pi}]$
 $C_n [e^{n\pi} - e^{-n\pi}] = \frac{2}{20} \int_0^{20} (20x - x^2) \sin\left(\frac{n\pi x}{20}\right) dx$
 $= \frac{1}{10} \left[\frac{20(20-x^2)}{n\pi} \cos\left(\frac{n\pi x}{20}\right) + \frac{400}{n^2 \pi^2} (20-2x) \sin\left(\frac{n\pi x}{20}\right) - \frac{16000}{n^3 \pi^3} \cos\left(\frac{n\pi x}{20}\right) \right]_0^{20}$

$$= \frac{1}{10} \left[\frac{-16000}{n^3 \pi^3} (-1)^n + \frac{16000}{n^3 \pi^3} (1) \right]$$

$$= \frac{16000}{n^3 \pi^3} [-(-1)^n + 1]$$

$$= \begin{cases} 0 & n=2, 4, 6, \dots \\ \frac{32000}{n^3 \pi^3} & n=1, 3, 5, \dots \end{cases}$$

$$C_n = \frac{32000}{n^3 \pi^3 (e^{n\pi} - e^{-n\pi})}$$

$$U(x, y) = \sum_{n=odd} \frac{32000}{n^3 \pi^3 (e^{n\pi} - e^{-n\pi})} \sin\left(\frac{n\pi y}{20}\right) (e^{\frac{n\pi x}{20}} - e^{-\frac{n\pi x}{20}})$$

A rectangular plate with insulated surfaces is 20cm wide and so long compared to its width that it may be considered infinite in length. If the temp. at the short edge $x=0$ is

$$U = \begin{cases} 10y & 0 \leq y \leq 10 \\ 10(20-y) & 10 \leq y \leq 20. \end{cases}$$

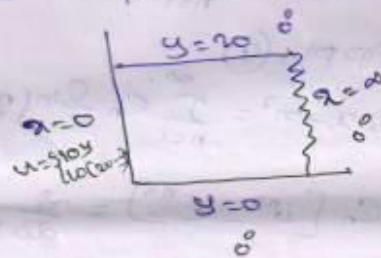
edges as well as the other short edge are kept at 0°C . Find the steady state temp. distribution in the plate.

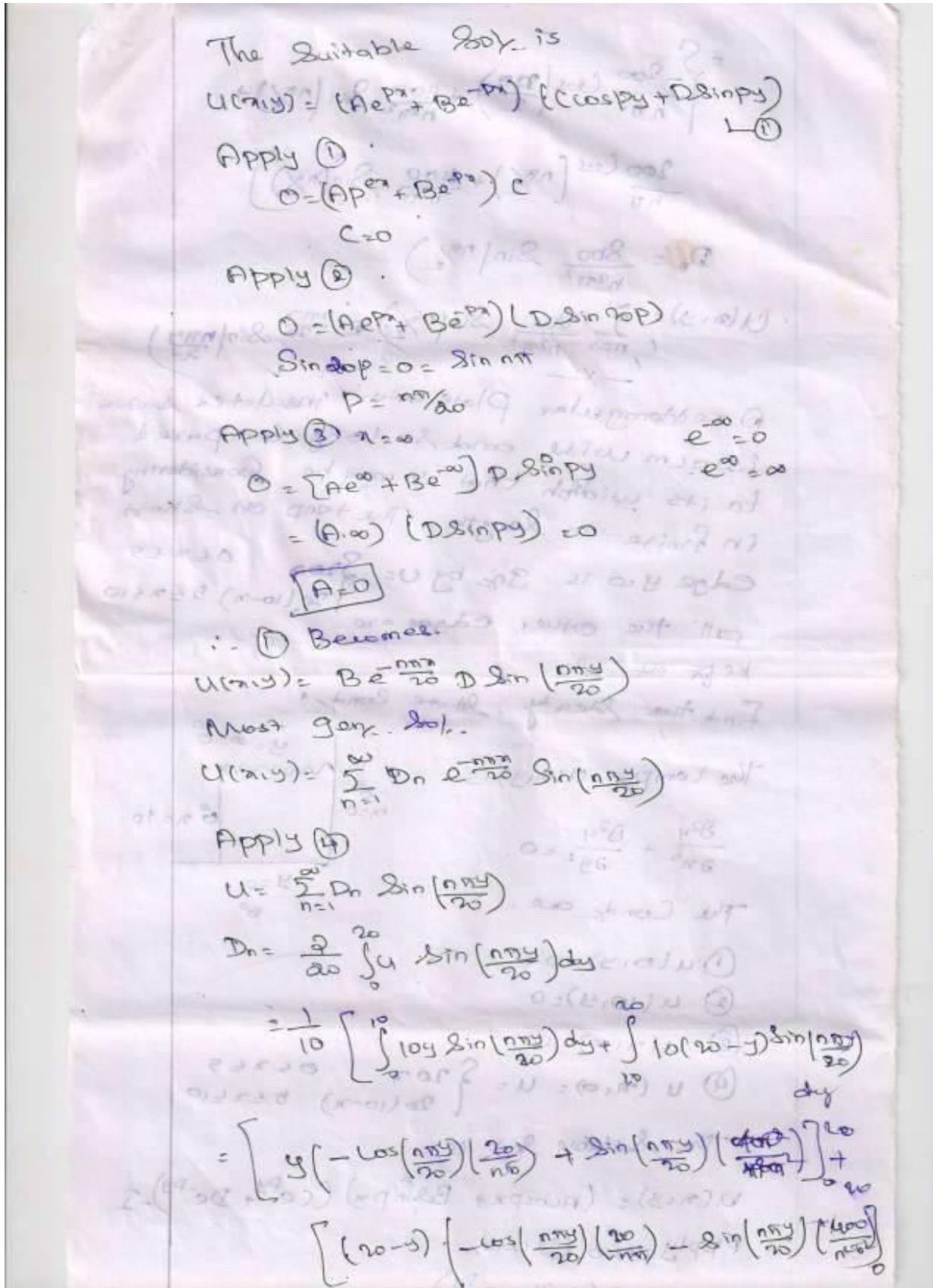
The temp $U(x, y)$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

The Cond. are.

- ① $U(x, 0) = 0$
- ② $U(x, 20) = 0$
- ③ $U(\infty, y) = 0$
- ④ $U(0, y) = U = \begin{cases} 10y & 0 \leq y \leq 10 \\ 10(20-y) & 10 \leq y \leq 20. \end{cases}$





$$= \int_{-\pi}^{\pi} \left[\frac{-200}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{400}{n^2\pi^2} \sin\left(n\frac{\pi}{2}\right) \right] + \dots$$

$$\frac{200}{n\pi} \left[\cos\left(n\frac{\pi}{2}\right) + \frac{400}{n\pi^2} \sin\left(n\frac{\pi}{2}\right) \right]$$

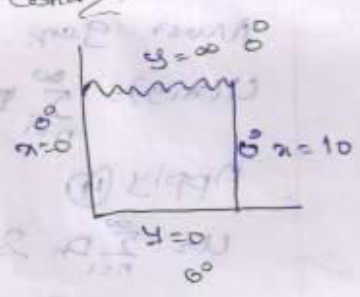
$$D_n = \frac{800}{n^2\pi^2} \sin\left(n\frac{\pi}{2}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2} \sin\left(n\frac{\pi}{2}\right) e^{-\frac{n\pi y}{20}} \sin\left(\frac{n\pi x}{20}\right)$$

A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considered finite in length. The temp at short edge $y=0$ is $3x$ by $u = \begin{cases} 20x & 0 \leq x \leq 5 \\ 20(10-x) & 5 \leq x \leq 10 \end{cases}$ all the other edges are kept at 0° .

Find the steady state cond.

The temp $u(x,y)$ is



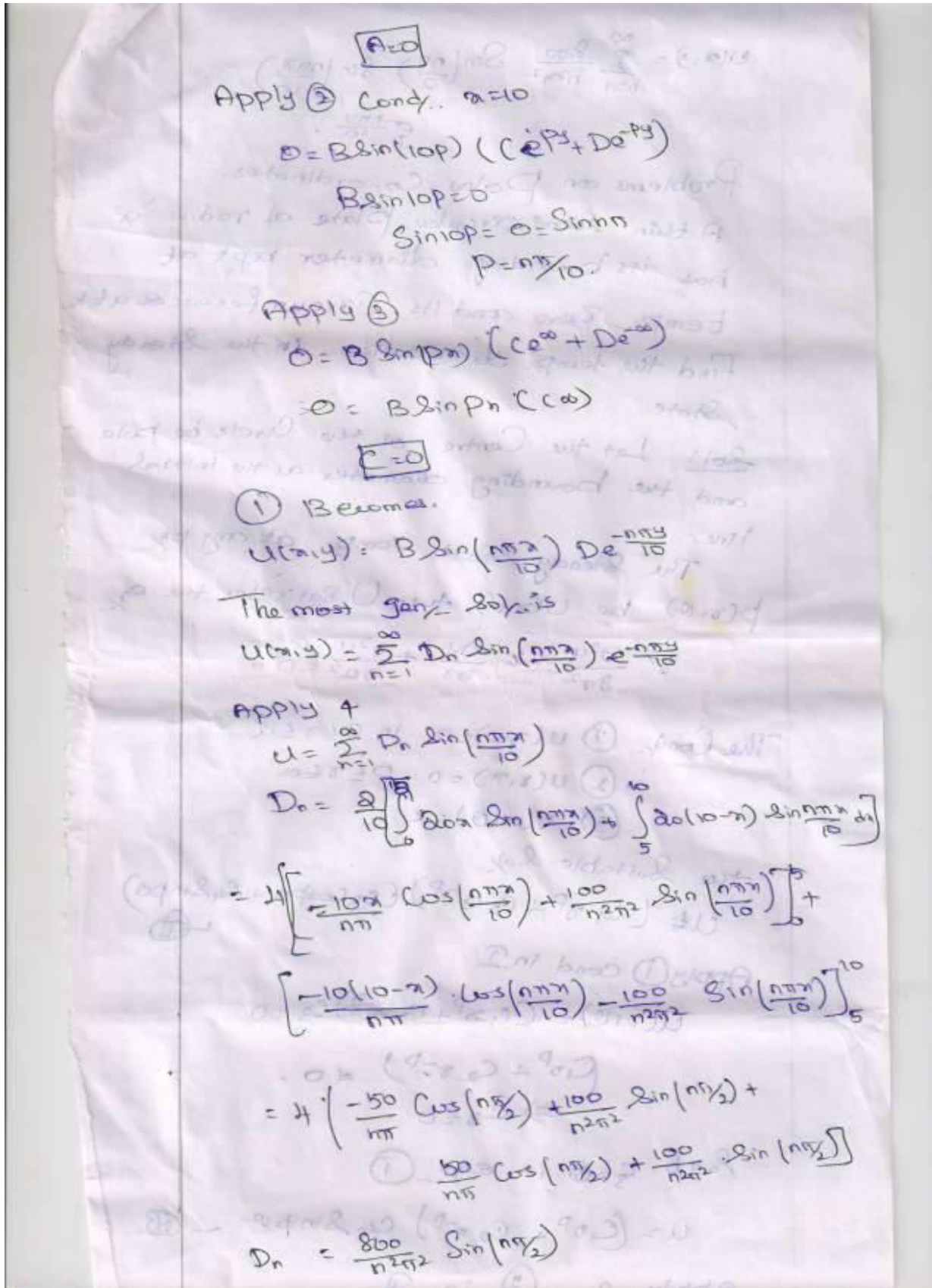
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Cond are:

- ① $u(0,y) = 0$
- ② $u(10,y) = 0$
- ③ $u(x,20) = 0$
- ④ $u(x,0) = u = \begin{cases} 20x & 0 \leq x \leq 5 \\ 20(10-x) & 5 \leq x \leq 10 \end{cases}$

The suitable soln is $u(x,y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py}) - 1$

Apply $x=0$ ① cond.



$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{B_{0n}}{n^2 \pi^2} \sin\left(\frac{n\theta}{2}\right) \sin\left(\frac{n\theta}{10}\right) e^{-\frac{n\pi y}{10}}$$

Problems on Polar, Co-ordinates.

A thin semi-circular plate of radius 'a' has its bounding diameter kept at temp. zero and its circumference at k. Find the temp. distribution in the steady state.

Soln Let the Centre of the Circle be pole and the bounding diameter as the initial line.

The steady state temp. at any pt. $P(r, \theta)$ be $U(r, \theta)$ then U satisfies the eqn.

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

The Cond. ① $u(r, 0) = 0$ in $0 \leq r \leq a$

② $u(r, \pi) = 0$ - $0 \leq r \leq a$

③ $u(a, \theta) = k$

the suitable soln.

$$u = (C_1 r^p + C_2 r^{-p}) (C_3 \cos p\theta + C_4 \sin p\theta) \quad \text{--- (1)}$$

Apply ① cond in ①

$$u(r, 0) = (C_1 r^p + C_2 r^{-p}) C_3 = 0$$

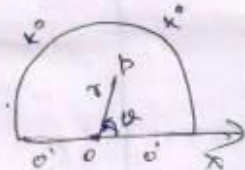
$$(C_1 r^p + C_2 r^{-p}) \neq 0$$

$$\Rightarrow C_3 = 0$$

Sub $C_3 = 0$ in eqn. ①

$$u = (C_1 r^p + C_2 r^{-p}) C_4 \sin p\theta \quad \text{--- (2)}$$

$u(x, \pi) = (C_1 r^p + C_2 r^{-p}) C_4 \sin p \pi = 0$
 $C_1 r^p + C_2 r^{-p} \neq 0, C_4 \neq 0 \Rightarrow \sin p \pi = 0$
 $\sin p \pi = 0 \Rightarrow \sin n \pi \quad \boxed{p = n}$
 Sub $p = n$ in ---
 $u = (C_1 r^n + C_2 r^{-n}) C_4 \sin n \theta$
 $u = 0 \text{ at } r = 0 \Rightarrow \boxed{C_2 = 0}$
 $(3) \Rightarrow u = C_1 r^n C_4 \sin n \theta$
 $u = C_n r^n \sin n \theta$



The most gen. soln
 $u(r, \theta) = \sum_{n=1}^{\infty} C_n r^n \sin n \theta$

Apply Cond. (3) in IV
 $u(a, \theta) = \sum_{n=1}^{\infty} C_n a^n \sin n \theta = k$
 $\Rightarrow \sum b_n \sin n \theta = k$ where $b_n = C_n a^n$
 $b_n = \frac{2}{\pi} \int_0^{\pi} k \sin n \theta d\theta$
 $= \frac{2k}{\pi} [-\cos n \theta]_0^{\pi} = \frac{-2k}{\pi n} [\cos n \pi - 1]$
 $= \frac{2k}{\pi n} [1 - \cos n \pi]$
 $C_n = \frac{1}{a^n} b_n = \frac{1}{a^n} \frac{2k}{\pi n} (1 - \cos n \pi)$
 $C_n = \begin{cases} 0 & n = \text{even} \\ \frac{4k}{a^n \pi} & n = \text{odd} \end{cases}$
 $u(r, \theta) = \frac{4k}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{r^n}{a^n} \sin n \theta$