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UNIT - I

LINEAR PROGRAMMING

Applications: LPP is used in many industrial & economic Problems. Its technique used for oil refineries, airlines, railways, textile industries, chemical industries, steel industries etc.

Defn: Linear Programming is the analysis of Problems in which a Linear function of a number of variables is to be optimized (Maximized or Minimized) when these variables are subject to a number of constraints in the Mathematical Linear inequalities.

General Form:

Maximize or minimize objective function

Subject to constraints.

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

(1) $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$, (or) $\ge$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$, (or) $\ge$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$ (or) $\ge$

(called structral constraints)

$x_1, x_2, x_3, \ldots, x_n \ge 0$

(called the non-negativity restrictions or constraints)

Note: Some of the constraints may be equalities. Some others may be inequalities of ($\le$) type and remaining ones inequalities of ($\ge$) type or all of them are of same type.

Defn: ① A set of values $x_1, x_2, \ldots x_n$ which satisfies the constraints of the LPP is called its Solution.

② Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible Solution.

③ Any feasible solution which optimizes (maximizes or minimizes) the objective function of the LPP is called its Optimum Solution or Optimal Solution.

④ If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, 3, \ldots k) \quad \text{--(i)}$$

then the non-negative variables S_i which are introduced to convert the inequalities (1) to the equalities $\sum_{j=1}^{n} a_{ij} x_j + S_i = b_i$ $(i = 1, 2, \ldots k)$ are called slack variables.

Max $\le$ slack +
min $\ge$

⑤ If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i \quad (i = k, k+1, \ldots) \quad \text{--(ii)}$$

then the non-negative variables S_i which are introduced to convert the inequalities ② to the equalities $\sum_{j=1}^{n} a_{ij} x_j - S_i = b_i$ $(i = k, k+1 \ldots)$ are called Surplus variables.

Canonical form of LPP:

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \le b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \le b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \le b_m$$

and the non-negativity restrictions

$$x_1, x_2, \ldots x_n \ge 0$$

This form of LPP is called the canonical form of the LPP.

Characteristics of the canonical form:

(i) The objective function is of Maximization type.

(ii) All constraints are of $(\le)$ type

(iii) All variables x_i are non-negative.

Standard form of LPP:

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to the constraints,

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

and $x_1, x_2, \ldots x_n \ge 0$ is known as standard form.

Characteristics of the standard form:

(i) The objective function is of maximization type.

(ii) All constraints are expressed as equations

(iii) Right hand side of each constraints is non-negative.

(iv) All variables are non-negative.

Note:

1) $\text{Min } f(x) = -\text{Max}(-f(x))$

$\text{Min } z = -\text{Max}(-z) = -\text{Max } z^*$

Ex: $\text{Min } z = c_1x_1 + c_2x_2$ is equivalent to

$\text{Max}(-z) = -c_1x_1 - c_2x_2$

2) An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (-1).

Ex: $ax_1 + bx_2 \geq c$

$\Rightarrow -ax_1 - bx_2 \leq -c$

3) If a variable is unconstrained or unrestricted (without specifying its sign). It can always be expressed as the difference of two non-negative variables.

Ex: If x_2 is unrestricted, then

$x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$

4) Whenever slack/surplus variables are introduced in the constraints, they should also appear in the objective function with zero coefficients.

Canonical form of LPP in matrix form:

1

Maximize $Z = CX$ (objective function)

Subject to $AX \leq b$ (constraints)

and $X \geq 0$ (non-negativity restrictions)

where $C = (c_1, c_2, \ldots c_n)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Standard form of LPP in matrix form:

Maximize $Z = CX$ (objective function)

Subject to constraints

$AX = b$ and $X \geq 0$

Formulation:

Step 1: Identify the decision variable

Step 2: Identify the constraints

Step 3: Identify the objective function.

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Problems based on formulation:

① A firm manufactures two types of products A, B and sells them at a profit of Rs 2 on type A and Rs 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the profit.

Solution:

Let the firm decide to produce x_1 units of product A and x_2 units of product B to maximize its profit.

Max → ≤, Min → ≥

To produce these units of type A and type B products, it requires

$x_1 + x_2$ processing minutes on M_1

$2x_1 + x_2$ processing minutes on M_2

∴ The complete formulation of the LPP is

$$\text{Maximize } Z = 2x_1 + 3x_2$$

Subject to the constraints

$$x_1 + x_2 \le 400$$
$$2x_1 + x_2 \le 600$$

and $x_1, x_2 \ge 0$.

② A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different type of foods. The yields per unit of these foods are given in the following table:

Food type	Yield/unit: Proteins	Fats	Carbohydrates	Cost/unit Rs
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the L.P. model for the problem.

Soln: Let x_1, x_2, x_3 and x_4 be the units of food of type 1, 2, 3 and 4 used respectively.

From these units of food of type 1, 2, 3 and 4 requires

$3x_1 + 4x_2 + 8x_3 + 6x_4$ proteins/day

$2x_1 + 2x_2 + 7x_3 + 5x_4$ fats/day

$6x_1 + 4x_2 + 7x_3 + 4x_4$ carbohydrates/day

The complete formulation of the L.P.P is

$$\text{Minimize } Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

Subject to
$$3x_1 + 4x_2 + 8x_3 + 6x_4 \ge 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \ge 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \ge 700 \text{ and}$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

(3) (Blending problem:)

A firm Produces an alloy having the following Specifications:

(i) Specific gravity ≤ 0.98

(ii) Chromium $\ge 8\%$

(iii) Melting Point $\ge 450^\circ C$

Raw Materials A, B and C having the properties Shown in the table can be used to make the alloy.

Property	Raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting Point	440°C	490°C	480°C

Cost of the various raw Materials Per unit ton are Rs 90 for A, Rs 280 for B and Rs 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Soln:- Let x_1, x_2 and x_3 be the tons of raw materials A, B and C to be used for making the alloy.

From these raw materials, the firm requires:

$0.92x_1 + 0.97x_2 + 1.04x_3$ Specific gravity

$7x_1 + 13x_2 + 16x_3$ Chromium

$440x_1 + 490x_2 + 480x_3$ Melting Point

$\therefore$ By the given Specifications, the constraints are

$$0.92x_1 + 0.97x_2 + 1.04x_3 \le 0.98$$

$$7x_1 + 13x_2 + 16x_3 \ge 8$$

$$440x_1 + 490x_2 + 480x_3 \ge 450$$

and $x_1, x_2, x_3 \ge 0$.

Since the cost of the various raw Materials Per unit ton are: Rs 90 for A, Rs 280 for B and Rs 40 for C, the total cost is $90x_1 + 280x_2 + 40x_3$

$\therefore$ The complete formulation of the LPP is

Minimize $z = 90x_1 + 280x_2 + 40x_3$.

Subject to $0.92x_1 + 0.97x_2 + 1.04x_3 \le 0.98$

$7x_1 + 13x_2 + 16x_3 \ge 8$

$440x_1 + 490x_2 + 480x_3 \ge 450$

and $x_1, x_2, x_3 \ge 0$.

(4) M.U. Old hens can be bought at Rs. 2 each and young ones at Rs 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs 1 per week to feed. A person has only Rs 80 to spend for hens. How many of each kind should he buy to give a profit of more than Rs 6 per week, assuming that he cannot house more than 20 hens. Formulate this as a L.P.P.

Soln: The person decides to buy x_1 old hens and x_2 young hens to Maximize his profit.

Since he has only Rs 80 to spend for hens and old hen costs Rs 2 and young hen costs Rs 5 each,

$$2x_1 + 5x_2 \le 80$$

Graphical Method :

Step 1 : Formulate the problem and define it is Simple Mathematical equations.

Step 2 : Plot the points and draw the lines accordingly.

Step 3 : Identify the solution area.

Techniques of Graphical Method :

To generate the solution mainly following two methods are used -

1. Corner point Method 2. Isoprofit Method.

Corner point Method :

Step 1 : Identify each of the extreme points or corner points of feasible region by the method of simultaneous equations.

Step 2 : calculate the profit at each of the corner pt.

Step 3 : a) The optimal solution occurs at that corner point which gives the highest profit in case of Maximization Problem.

b) The optimal solution occurs at that corner point which gives the lowest profit in case of Minimization Problem.

Isoprofit Method [Iso-cost Method]

It shows that any set of combinations of points produce the same profit as any other combination on the same line.

Step 1 : Draw an isoprofit line within the constraints of Problems. It is a straight line on which every point has same profit.

Problems based ...

1. Solve the following LPP graphically

Maximize $Z = 5x_1 + 3x_2$

Subject to $3x_1 + 5x_2 \le 15$, $5x_1 + 2x_2 \le 10$, $x_1, x_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equations

From the given LPP, the constraints are converted into equations as follows:

$$3x_1 + 5x_2 = 15 \rightarrow (1)$$
$$5x_1 + 2x_2 = 10 \rightarrow (2)$$

Step 2: To find ordinates:

From (1), let $x_1 = 0$ then $5x_2 = 15$

$x_2 = 3$

let $x_2 = 0$ then $3x_1 = 15$

$x_1 = 5$

$\therefore$ ordinates are $(0,3)$ and $(5,0)$

From (2), If $x_1 = 0$ then $2x_2 = 10$

$x_2 = 5$

and If $x_2 = 0$ then $5x_1 = 10 \Rightarrow x_1 = 2$

$\therefore$ ordinates are $(0,5)$ and $(2,0)$

Step 3: Graphical representation of given LPP using ordinates is shown below:

Sub $x_2 = \frac{45}{19}$ in eqn ② $5x_1 + 2x_2 = 10$

$$5x_1 + 2\left(\frac{45}{19}\right) = 10$$

$$\Rightarrow 5x_1 = 10 - \frac{90}{19}$$

$$\Rightarrow 5x_1 = \frac{100}{19}$$

$$\Rightarrow x_1 = \frac{100}{5 \times 19} = \frac{20}{19}$$

$$x_1 = \frac{20}{19}$$

$\therefore$ vertex of B is $\left(\frac{20}{19}, \frac{45}{19}\right)$

The values of objective function z at these vertices are given by

Vertex	Max $z = 5x_1 + 3x_2$
$O(0,0)$	0
$A(2,0)$	10
$B\left(\frac{20}{19}, \frac{45}{19}\right)$	$\frac{235}{19}$
$C(0,3)$	9

Here, Maximum value of Z occurs at the vertex B.

$\therefore$ The optimal solution is

Maximum $Z = \frac{235}{19}$ at $x_1 = \frac{20}{19}$ and $x_2 = \frac{45}{19}$

② Solve the following L.P.P by the graphical

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to $-2x_1 + x_2 \le 1$

$x_1 \le 2$

$x_1 + x_2 \le 3$ and $x_1, x_2 \ge 0$

Soln:

Step 1: First consider the inequality constraints as equalities.

$-2x_1 + x_2 = 1 \longrightarrow$ ①

$x_1 = 2 \longrightarrow$ ②

$x_1 + x_2 = 3 \longrightarrow$ ③

and $x_1 = 0$, $x_2 = 0$.

Step 2: To find ordinates:

From eqn ①, let $x_1 = 0 \Rightarrow x_2 = 1$

let $x_2 = 0 \Rightarrow -2x_1 = 1$

$x_1 = -0.5$

∴ ordinates are $(0,1)$ and $(-0.5, 0)$

From eqn ③

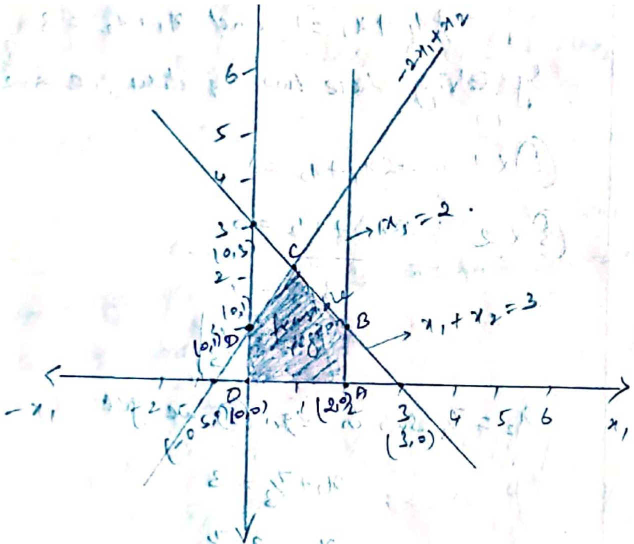
Put $x_1 = 0$ $x_2 = 3$

Put $x_2 = 0$ $x_1 = 3$

∴ ordinates are $(0,3)$ and $(3,0)$

From ②, $x_1 = 2$

Step 3: Graphical representation of given LPP using [ord]inates is shown below.



By using Graphical Method, the solution space OABCD shown shaded in the above figure.

∴ The vertices of this region are $O(0,0)$, $A(2,0)$ and $D(0,1)$

To find the vertex B:

B is the intersection point of the two lines

$x_1 + x_2 = 3$ ③ and $\boxed{x_1 = 2} \rightarrow$ ②

Eqn ② sub in eqn ③ we get $2 + x_2 = 3 \Rightarrow x_2 = 3 - 2 \Rightarrow \boxed{x_2 = 1}$

∴ vertex of B is $(2,1)$

To find the vertex C

C is the intersection of the two lines

(3) Apply graphical [illegible]

$Z = x_1 - 2x_2$ Subject to $-x_1 + x_2 \leq 1$, $6x_1 + 4x_2 \geq 24$

$0 \leq x_1 \leq 5$ and $2 \leq x_2 \leq 4$.

U.Q.

Solution:

Step 1: convert the inequalities into equations.

From the given LPP, the constraints are converted into equations as follows:

$-x_1 + x_2 = 1 \longrightarrow (1)$

$6x_1 + 4x_2 = 24 \longrightarrow (2)$

$x_1 = 0$, $x_1 = 5$ and $x_2 = 2$, $x_2 = 4$

Step 2: To find ordinates:

from (1), Put $x_1 = 0$ then $x_2 = 1$

$x_2 = 0$ then $x_1 = -1$

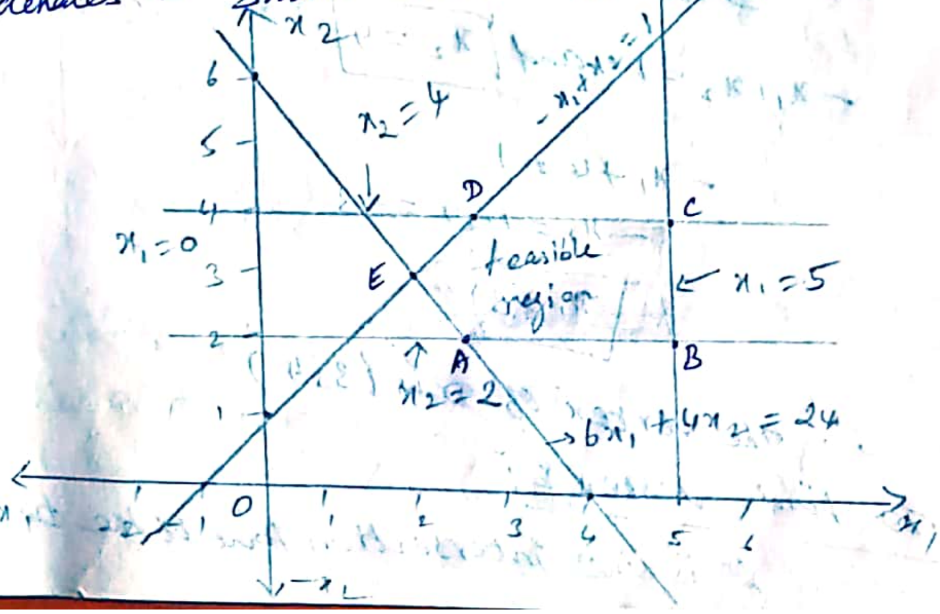
$\therefore$ ordinates are $(0,1)$ and $(-1,0)$

from (2), $x_1 = 0$ then $4x_2 = 24$
$x_2 = 6$

$x_2 = 0$ then $6x_1 = 24$
$x_1 = 4$

$\therefore$ ordinates are $(0,6)(4,0)$

Step 3: Graphical representation of given LPP using ordinates is shown below:



$(m<n)$. The solution obtained by setting $(n-m)$ variables equal to zero and solving for the remaining m variables is called a basic solution.

The m variables are called basic variables and they form the basic solution. The $(n-m)$ variables which are put to zero are called as non-basic variables.

2) A basic solution is said to be a non-degenerate basic solution if none of the basic variables is zero.

3) A basic solution is said to be a degenerate basic solution if one or more of the basic variables are zero.

4) A feasible solution which is also basic is called a basic feasible solution.

① Find all the basic solutions to the following problem

Maximize $Z = x_1 + 3x_2 + 3x_3$ subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$. Also find which of the basic solutions are (i) basic feasible (ii) non-degenerate basic feasible (iii) optimal basic feasible.

Soln. Since there are $m=2$ equations with $n=3$ variables, the basic solutions are obtained by setting $(n-m) = (3-2) = 1$ variable equal to zero and solving for the remaining two variables. Since there are 3 variables with 2 eqns we shall have $3C_2 = 3$ different basic solutions

$2x_1 + 3x_2 = 7$

$x_1 = 2, x_2 = 1$

2.	x_1, x_3	$x_2 = 0$	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = -1, x_3 = 1$	4	No	Yes
3.	x_2, x_3	$x_1 = 0$	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	3	No	Yes

$\therefore$ The optimal soln is Max $z = 5$, $x_1 = 2$, $x_2 = 1$, $x_3 = 0$

Variants of the Simplex method

$(x_1, x_2, x_3$ any) atleast one variable is zero

1. Degeneracy and cycling → Repeated negative in the table $z_j - c_j$
2. unbounded solution → Graphical method open region
3. Multiple Solutions,
4. Non-existing feasible solution → Big M method
5. unrestricted variables. Hence

Problems solved

Maximize $Z = 4x_1 + 10x_2$ Max $\Rightarrow \leq$ -

Subject to $2x_1 + x_2 \leq 50$ Min $\geq$ - surplus

3 eqns means 3 slack variable introduced

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$ and $x_1, x_2 \geq 0$

Suppose 2 eqns - 2 slack variable

Solution:

Step 1: By introducing the slack variables S_1, S_2 and S_3, the problem is standard form becomes

Maximize $Z = 4x_1 + 10x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$2x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 50$

$2x_1 + 5x_2 + 0S_1 + S_2 + 0S_3 = 100$

$2x_1 + 3x_2 + 0S_1 + 0S_2 + S_3 = 90$ and

$x_1, x_2, S_1, S_2, S_3 \geq 0$.

Step 2: Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5-3) = 2$ variables to zero.

$\therefore$ The Initial basic feasible solution is $S_1 = 50, S_2 = 100, S_3 = 90$

$x_1 = 0, x_2 = 0$ (non basic)

$S_1 = 50, S_2 = 100, S_3 = 90$

Step 3: ... simplex table is given by

$10\ x_2\ 20\ \ 2/5\ \ 1\ \ 0\ \ 1/5\ \ 0$ 20, which corresponds to S_2

$0\ S_3\ 30\ \ 4/5\ \ 0\ \ 0\ \ -3/5\ \ 1$

$Z_j - C_j\ \ 200\ \ 0\ \ 0$

$\therefore$ The leaving variable is the basic variable S_2 which corresponds to the minimum ratio $\theta = 20$.

The leaving variable row is called the pivot-row (or) key row (or) pivot equation and $\underline{\underline{5}}$ is the pivot element.

<u>Step 5</u> : Now, New pivot equation = old pivot eqn $(\div)$ pivot element

$$\Rightarrow (100\quad 2\quad 5\quad 0\quad 1\quad 0)/5$$

$$\Rightarrow 20\quad 2/5\quad 1\quad 0\quad 1/5\quad 0 \rightarrow \text{New pivot eqn}$$

New S_1 equation = old S_1 eqn $-$ (corresponding column coefficient) $\times$ (New pivot eqn)

$$\begin{array}{rrrrrrl} = 50 & 2 & 1 & 1 & 0 & 0 & \\ 20 & 2/5 & 1 & 0 & 1/5 & 0 & (-) \\ \hline 30 & 8/5 & 0 & 1 & -1/5 & 0 & \end{array}$$

New S_3 equation = old S_3 eqn $-$ (3) $\times$ New pivot eqn

$$\begin{array}{rrrrrrl} = 90 & 2 & 3 & 0 & 0 & 1 & \\ 60 & 6/5 & 3 & 0 & 3/5 & 0 & (-) \\ \hline 30 & 4/5 & 0 & 0 & -3/5 & 1 & \end{array}$$

New $(Z_j - C_j)$ eqn = old $(Z_j - C_j)$ eqn $-$ (-10) New pivot eqn

$$\begin{array}{rrrrrrl} = 0 & -4 & -10 & 0 & 0 & 0 & \\ -200 & -4 & -10 & 0 & -2 & 0 & (-) \\ \hline 200 & 0 & 0 & 0 & 2 & 0 & \end{array}$$

∴ The improved basic feasible solution is given in the following simplex method.

New row = current row − Pivot column coefficient × New Pivot row

<u>First Iteration</u>: Introduce x_2 and drop s_2

		C_j	4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	30	8/5	0	1	−1/5	0
10	x_2	20	2/5	1	0	1/5	0
0	S_3	30	4/5	0	0	−3/5	1
	$Z_j - C_j$	200	0	0	0	2	0

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

all values are +ve then the problem

∴ The optimal solution is

Max $Z = 200$ at $x_1 = 0$, $x_2 = 20$

② Solve the following

Maximize : $15x_1 + 6x_2 + 9x_3 + 2x_4$

Subject to $2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$

$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$

$7x_1 + x_4 \leq 70$

$x_1, x_2, x_3, x_4 \geq 0$.

<u>Soln</u>: By introducing non-negative slack variables S_1, S_2 and S_3 the standard form of the LPP becomes. Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3$

Subject to $2x_1 + x_2 + 5x_3 + 6x_4 + S_1 + 0S_2 + 0S_3 = 20$

$3x_1 + x_2 + 3x_3 + 25x_4 + 0S_1 + S_2 + 0S_3 = 24$

$7x_1 + 0x_2 + 0x_3 + x_4 + 0S_1 + 0S_2 + S_3 = 70$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$

Since there are 3 eqns with 7 variables the initial basic feasible soln is obtained equating (7−3) = 4 variables to zero

∴ The initial basic feasible solution is $S_1 = 20$

$S_2 = 24$, $S_3 = 70$. ($x_1 = x_2 = x_3 = x_4 = 0$ non basic)

The initial simplex table is given by

<u>Initial Iteration</u>:

		C_j	15	6	9	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	$\theta = \min \frac{X_B}{x_1}$
0	S_1	20	2	1	5	6	1	0	0	10
0	S_2	24	(3)	1	3	25	0	1	0	8*
0	S_3	70	7	0	0	1	0	0	1	10
$Z_j - C_j$		0	−15	−6	−9	−2	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable S_2 leaves the basis.

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15	x_1	8	1	$\frac{1}{3}$	1	$\frac{25}{3}$	0	$\frac{1}{3}$	0	
0	S_3	14	0	$-\frac{7}{3}$	-7	$-\frac{172}{3}$	0	$-\frac{7}{3}$	+1	
$Z_j - C_j$		~~0~~	~~-15~~	~~-6~~	~~-9~~	~~-2~~	~~0~~	~~0~~	~~0~~	
		120	0	-1	6	123	0	5	0	

$6 - \frac{50}{3}$

$\frac{18 - 50}{3}$

$70 - 56$

New Pivot equation = old Pivot eqn ÷ Pivot element

$= (24 \quad 3 \quad 1 \quad 3 \quad 25 \quad 0 \quad 1 \quad 0)/3$

$= 8 \quad 1 \quad \frac{1}{3} \quad 1 \quad \frac{25}{3} \quad 0 \quad \frac{1}{3} \quad 0$

New S_1 eqn = old S_1 eqn − 2 (New Pivot eqn)

$= 20 \quad 2 \quad 1 \quad 5 \quad 6 \quad 1 \quad 0 \quad 0$

$ 16 \quad 2 \quad \frac{2}{3} \quad 2 \quad \frac{50}{3} \quad 0 \quad \frac{2}{3} \quad 0 \quad (-)$

$ 4 \quad 0 \quad \frac{1}{3} \quad 3 \quad -\frac{32}{3} \quad 1 \quad -\frac{2}{3} \quad 0$

New S_3 eqn = old S_3 eqn − 7 (New Pivot eqn)

$= 70 \quad 7 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1$

$ 56 \quad 7 \quad \frac{7}{3} \quad 7 \quad \frac{175}{3} \quad 0 \quad \frac{7}{3} \quad 0 \quad (-)$

$ 14 \quad 0 \quad -\frac{7}{3} \quad -7 \quad -\frac{172}{3} \quad 0 \quad -\frac{7}{3} \quad +1$

New x_3 eqn

$= 14 \quad 0 \quad -\frac{7}{3} \quad -7 \quad -\frac{172}{3} \quad 0 \quad -\frac{7}{3}$

$-28 \quad 0 \quad -\frac{7}{3} \quad -21 \quad \frac{224}{3} \quad -7 \quad \frac{14}{3} \quad 0$ (−)

$42 \quad 0 \quad 0 \quad 14 \quad -132 \quad 7 \quad -7$

New $(z_j - c_j)$ = old $(z_j - c_j)$ eqn $- (-1)$ New Pivot eqn

$= 120 \quad 0 \quad -1 \quad 6 \quad 123 \quad 0 \quad 5 \quad 0$

$-12 \quad 0 \quad -1 \quad -9 \quad 32 \quad -3 \quad 2 \quad 0$ (−)

$132 \quad 0 \quad 0 \quad 15 \quad 91 \quad 3 \quad 3 \quad 0$

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is given by

Max $z = 132$, $x_1 = 4$, $x_2 = 12$, $x_3 = 0$ and $x_4 = 0$.

New x_1 eqn = old x_1 eqn $- (\frac{1}{3})$ (New Pivot eqn)

$= 8 \quad 1 \quad \frac{1}{3} \quad 1 \quad \frac{25}{3} \quad 0 \quad \frac{1}{3} \quad 0$

$4 \quad 0 \quad \frac{1}{3} \quad 3 \quad -\frac{32}{3} \quad 1 \quad -\frac{2}{3} \quad 0$ (−)

$4 \quad 1 \quad 0 \quad -2 \quad \frac{57}{3} \quad -1 \quad 1 \quad 0$

(3) Maximize $z = 3x_1 + 2x_2 + 5x_3$

Subject to $x_1 + 4x_2 \leq 420$

$3x_1 + 2x_3 \leq 460$

$x_1 + 2x_2 + x_3 \leq 430$

First Iteration, Second Iteration.

Ans: Max $z = 1350$, $x_1 = 0$ $x_2 = 100$ $x_3 = 230$

(4) Solve the following LPP by Simplex Method.

Minimize $z = 8x_1 - 2x_2$

Subject to $-4x_1 + 2x_2 \leq 1$

$5x_1 - 4x_2 \leq 3$ and $x_1, x_2 \geq 0$.

Solution:

Step 1: Since the given objective function is of Minimization type, we shall convert it into a Maximization type as follows.

Maximize $(-z)$ = Maximize $z^* = -8x_1 + 2x_2$

Subject to $-4x_1 + 2x_2 \leq 1$

$5x_1 - 4x_2 \leq 3$

$x_1, x_2 \geq 0$

By introducing non-negative slack variables S_1, S_2 the standard form of the LPP becomes

Maximize $z^* = -8x_1 + 2x_2 + 0S_1 + 0S_2$

Subject to the constraints

$-4x_1 + 2x_2 + S_1 + 0S_2 = 1$

$5x_1 - 4x_2 + 0S_1 + S_2 = 3$

and $x_1, x_2, S_1, S_2 \geq 0$

Two Phase Method:

(1) The two phase method is another method to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows.

Phase I : In this phase, the simplex method is applied to a specially constructed auxiliary LPP leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1: Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function. Thus the new objective function is $z^* = -R_1 - R_2 - R_3 - \dots - R_n$ where R_i's are the artificial variables.

Step 2: Construct the auxiliary LPP in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3: Solve the auxiliary LPP by simplex method until either of the following three possibilities arise.

(i) Max $z^* < 0$ and atleast one artificial variable appears in the optimum basis at a non-zero level. In this case the given LPP does not posses any feasible solution. Stop the procedure.

(ii) Max $z^* = 0$ and atleast one artificial variable appears in the optimum basis at zero level. In this case proceed to Phase-II

(iii) Max $z^* = 0$ and no artificial variable appears in the optimum basis. In this case proceed to phase-II

Phase-II . Use the optimum basic feasible solution of Phase-I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a c... to every artificial variable that appears in the basis at the zero level. Use simplex method to the modified simplex table obtained at the end of Phase-I, till an optimum basic feasible solution (if any) is obtained.

Note 1: In Phase-I, the iterations are stopped as soon as the value of the new objective function becomes zero because this is its maximum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.

Note 2: The new objective function is always of maximization type regardless of whether the original problem is of maximization or minimization type.

Note 3: Before starting Phase-II, remove all artificial variables from the table which were non-basic at the end of Phase-I.

Problems:

(1) Use Two-Phase Simplex Method to solve

Maximize $Z = 5x_1 + 8x_2$.

Subject to constraints
$$3x_1 + 2x_2 \geq 3$$
$$x_1 + 4x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0.$$

(−) surplus; greater than — Artificial element add; + slack.

Solution:

By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to
$$3x_1 + 2x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 3$$
$$x_1 + 4x_2 + 0s_1 - s_2 + 0s_3 + R_2 = 4$$
$$x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 5$$

and $x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0$.

(Here s_1, s_2 – surplus, s_3 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 3$, $R_2 = 4$, $s_3 = 5$ (basic) $(7-3=4)$

$(x_1 = x_2 = s_1 = s_2 = 0$, non basic$)$

Phase-I : Assigning a cost −1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } z^* = -R_1 - R_2$$

Subject to the given constraints.

The iterative simplex tables for the auxiliary new R_1 $\left(1 \quad 5/2 \quad 0 \quad -1 \quad 1/2 \quad 0 \quad 1 \quad -1/2\right) / (5/2)$

Second Iteration: Introduce x_1 and drop R_1.

C_j (0 0 0 0 0 −1 −1)

	C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	R_1	R_2
new pivot	0	x_1	$2/5$	1	0	$-2/5$	$1/5$	0	$2/5$	$-1/5$
old $x_2 - (1/4)$ new pivot	0	x_2	$9/10$	0	1	$1/10$	$-3/10$	0	$-1/10$	$3/10$
old $s_3 - (3/4)$ new pivot eqn	0	S_3	$37/10$	0	0	$3/10$	$1/10$	1	$-3/10$	$-1/10$
old $z_j^* - c_j - (-5/2)$ new pivot		$Z_j^* - C_j$	0	0	0	0	0	0	1	1

all values are +ve

this place is same as direct stop the process

Since all $(z_j^* - c_j) \geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so we proceed to Phase-II

we're to continue optimal, & artificial variable exists (is zero)

Phase-II Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0S_1 + 0S_2 + 0S_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase-I.

The iterative simplex tables for this phase are:

Initial Iteration: C_j (5 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	$\theta = X_B / S_2$
5	x_1	$2/5$	1	0	$-2/5$	$(1/5)$	0	2^*
8	x_2	$9/10$	0	1	$1/10$	$-3/10$	0	—
0	S_3	$37/10$	0	0	$3/10$	$1/10$	1	37
	$Z_j - C_j$	$46/5$	0	0	$-6/5$	$-7/5$	0	

$(5 \; 8 \; 0)\begin{pmatrix} 2/5 \\ 9/10 \\ 37/10 \end{pmatrix}$

$2 + \frac{9 \times 8}{10} + 0 = 2 + \frac{36}{5} = \frac{46}{5}$

$(5 \; 8 \; 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 5 = 5 - 5 = 0$

$(z_j - c_j)$	12	7	0	-4	0	0	

old $z_j - c_j = (-7/5)$ new pivot eqn

Since there are some $(z_j - c_j) < 0$, current basic feasible solution is not optimal.

new pivot 7 -1 0 1 0 2

<u>Second</u> <u>Iteration</u> : Introduce S_1 and drop S_3

		c_j	5	8	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_2	16	3	0	0	1	4
8	x_2	5	1	1	0	0	1
0	S_1	7	-1	0	1	0	2
	$(z_j - c_j)$	40	3	0	0	0	8

old $S_2 - (-2)$ new pivot

old $x_2 - (-1/2)$ new pivot

old $z_j - c_j - (-4)$ new pivot eqn

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 40$, $x_1 = 0$, $x_2 = 5$

Sensitivity Analysis

I. Variation in the right side of constraints :

1. consider the LPP Max $z = 5x_1 + 12x_2 + 4x_3$

Subject to $x_1 + 2x_2 \le 5$, $5x_1 - x_2 + 2x_3 = 2$ and $x_1, x_2, x_3 \geq 0$.

a) Solve the LPP.

b) Discuss the effect of changing the requirement vector from ~~(5/2) to (4/2)~~ $\begin{pmatrix}5\\2\end{pmatrix}$ to $\begin{pmatrix}7\\2\end{pmatrix}$, on the optimal solution.

Solution: a) By using Big-M method, the optimal simplex table is displayed below.

		c_j	5	12	4	0	−M
C_B	y_B	x_B	x_1	x_2	x_3	s_1	R_1
12	x_2	5/2	1/2	1	0	1/2	0
4	x_3	9/4	11/4	0	1	1/4	1/2
$z_j - c_j$		39	12	0	0	7	M+2

$\therefore$ The optimal solution is Max $z = 39$, $x_1 = 0$, $x_2 = 5/2$, $x_3 = 9/4$.

b) If the requirement vector ~~(5/2)~~ $\begin{pmatrix}5\\2\end{pmatrix}$ is changed to $\begin{pmatrix}7\\2\end{pmatrix}$ then $x_B = B^{-1}b = \begin{pmatrix}1/2 & 0\\ 1/4 & 1/2\end{pmatrix}\begin{pmatrix}7\\2\end{pmatrix}$

$$= \begin{pmatrix}7/2\\ 11/4\end{pmatrix} = \begin{pmatrix}x_2\\ x_3\end{pmatrix}$$

Since both $x_2 \geq 0$ and $x_3 \geq 0$, the current solution consisting x_2, x_3 remains feasible. But the value of x_2 and x_3 are changed.

Now optimal solution is Max $z = 5x_1 + 12x_2 + 4x_3$

$= 5(0) + 12(7/2) + 4(11/4)$

$= 53$

2. Consider the LPP Max $z = 2x_1 + x_2 + 4x_3 - x_4$

Subject to $x_1 + 2x_2 + x_3 - 3x_4 \le 8$

$-x_2 + x_3 + 2x_4 \le 0$

$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$ and $x_1, x_2, x_3, x_4 \ge 0$.

a) Solve the LPP

b) Discuss the effect of change of b_2 to 11.

c) Discuss the effect of change of b to [3 -2 4]

Solution:

a) By using regular simplex method, the optimal simplex table is displayed below.

	c_j	(2	1	4	-1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
1	x_2	0	0	1	-1	-2	0	-1	0
2	x_1	8	1	0	3	1	1	2	0
0	S_3	5	0	0	-4	2	-2	3	1
$Z_j - C_j$		16	0	0	1	1	2	3	0

$\therefore$ The optimal solution is

Max $z = 16$, $x_1 = 8$, $x_2 = x_3 = x_4 = 0$

b) If b_2 is changed to 11, then $b = [8\ \ 11\ \ 21]$

then $X_B = B^{-1}b = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \\ 21 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \\ 38 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ S_3 \end{pmatrix}$

$\Rightarrow$ the solution is infeasible.

i.e., the change of b_2 from 0 to 11 affects the feasibility of the solution.

The changes in the final table reduce to

	c_j	(2	1	4	-1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
1	x_2	-11	0	1	-1	-2	0	-1	0
2	x_1	30	1	0	3	1	1	2	0
0	S_3	38	0	0	-4	2	-2	3	1
$Z_j - C_j$		49	0	0	1	1	2	3	0

Since the current solution is infeasible, we have to use dual simplex method.

First Iteration: Drop x_2 and introduce x_4.

	c_j	(2	1	4	-1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
-1	x_4	$11/2$	0	$-1/2$	$1/2$	1	0	$1/2$	0
2	x_1	$\frac{49}{2}$	1	$1/2$	$5/2$	0	1	$3/2$	0
0	S_3	27	0	1	-5	0	-2	2	1
$Z_j - C_j$		$\frac{87}{2}$	0	$1/2$	$1/2$	0	2	$5/2$	0

Since all $(Z_j - C_j) \ge 0$ and $X_{Bi} \ge 0$, the current solution is feasible and optimal.

$\therefore$ The optimal solution to the new problem is

Max $z = \frac{87}{2}$, $x_1 = 49/2$, $x_2 = 0$, $x_3 = 0$, $x_4 = 11/2$

c) If b is changed from (8 0 21) to (3 -2 4)

Then $X_B = B^{-1}b \Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ S_3 \end{pmatrix}$

$\Rightarrow$ the solution is infeasible.

The changes in the final table reduce to

2	x_1	-1	1	0	3	1	1	2	0
0	s_3	-8	0	0	(-4)	2	-2	3	1
					$-1/4$	$1/2$	-1	1	0
$z_j - c_j$		0	0	0	1	1	2	3	0

Since the solution is infeasible, we have to use dual simplex method.

<u>First Iteration</u> : Drop s_3 and introduce x_3.

		c_j	2	1	4	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_2	4	0	1	0	$-5/2$	$1/2$	$-7/4$	$-1/4$
2	x_1	-7	1	0	0	$5/2$	$(-1/2)$	$\frac{17}{4}$	$3/4$
4	x_3	2	0	0	1	$-1/2$	$1/2$	$-3/4$	$-1/4$
$z_j - c_j$		-2	0	0	0	$3/2$	$3/2$	$15/4$	$1/4$

<u>Second Iteration</u>: Drop x_1 and introduce s_1

		c_j	2	1	4	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_2	-3	1	1	0	0	0	$5/2$	$1/2$
0	s_1	14	-2	0	0	-5	1	$-17/4$	$-3/2$
4	x_3	-5	1	0	1	2	0	$7/2$	$1/2$
$z_j - c_j$		-23	3	0	0	9	0	$\frac{33}{2}$	$5/2$

Since x_3 leaves the basis and all $a_{3j} \geq 0$ the new problem possess no feasible solution.