UNIT - II

Duality and Networks :

Formulation of dual problems :

Symmetric form :

Given primal $\rightarrow$ dual

x	y
Max $\leq$	
Min $\geq$	

Consider the following LPP

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$

(ii) The maximization problem has ($\leq$) constraints while the minimization problem has ($\geq$) constraints.

(iii) If the primal contains m constraints and n variables, then the dual will contain n constraints and m variables. i.e., the transpose of the body matrix of the primal problem gives the body matrix of the dual and vice versa.

(iv) The constants $c_1, c_2, c_3, \dots c_n$ in the objective function of the primal appears in the constraints of the dual.

(v) The constants $b_1, b_2, \dots b_m$ in the constraints of the primal appears in the objective function of the dual.

(vi) The variables in both problems are non-negative.

Primal variables

$$\begin{array}{c|c|c}
 & x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n & \\
y_1 & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} & \begin{matrix} \leq b_1 \\ \leq b_2 \\ \vdots \\ \leq b_m \end{matrix} \\
y_2 & & \\
\vdots & & \\
y_m & & \\
 & \geq \quad \geq \quad \geq \quad \dots \quad \geq & \\
 & c_1 \quad c_2 \quad c_3 \quad \dots \quad c_n &
\end{array}$$

Dual variables

R.H.S of primal constraints

R.H.S of dual constraints.

Then the dual problem of eqn (1) will be

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$

Minimize $W = b^T y$

Subject to the constraints $A^T y \geq c^T$ and $y \geq 0$

i.e. Minimize $W = b_1 y_1 + b_2 y_2 + \dots + b_m y_m$

Subject to

$$\left.\begin{aligned} a_{11} y_1 + a_{21} y_2 + \dots + a_{m1} y_m &\geq c_1 \\ a_{12} y_1 + a_{22} y_2 + \dots + a_{m2} y_m &\geq c_2 \\ &\vdots \\ a_{1n} y_1 + a_{2n} y_2 + \dots + a_{mn} y_m &\geq c_n \\ \text{and } y_1, y_2, \dots y_m &\geq 0 \end{aligned}\right\} \to (2)$$

Eqns (1) & (2) are called symmetric primal-dual pairs.

(1) Write the dual of the following primal LPP.

Maximize $F = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 - x_3 \leq 2$$
$$-2x_1 + x_2 - 5x_3 \geq -6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0.$$

Soln: Given primal LPP

Maximize $F = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6 \text{ and } x_1, x_2, x_3 \geq 0.$$

Max $F = CX$

Subject to $AX \leq b$ and $X \geq 0$.

where $C = (1 \ 2 \ 1)$, $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$

and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal problem is maximization type with ($\leq$) type constraints, with 3 constraints and 3 variables, the dual problem is minimization type with ($\geq$) type constraints, with 3 constraints and 3 dual variables y_1, y_2, y_3.

The dual problem is minimize $W = b^T y$

Subject to the constraints $A^T y \geq C^T$

and $y \geq 0$.

i.e, $\text{Min } W = \begin{pmatrix} 2 & 6 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

S. to $\begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$

i.e, $\text{Min } w = 2y_1 + 6y_2 + 6y_3$

Subject to $2y_1 + 2y_2 + 4y_3 \geq 1$

$y_1 - y_2 + y_3 \geq 2$

$-y_1 + 5y_2 + y_3 \geq 1$

and $y_1, y_2, y_3 \geq 0$.

② Find the dual of the LPP

$\text{Max } z = 3x_1 - x_2 + x_3$

Max Subject to $4x_1 - x_2 \leq 8$

min $8x_1 + x_2 + 3x_3 \geq 12$

$5x_1 - 6x_3 \leq 13$ and $x_1, x_2, x_3 \geq 0$.

Soln:

Given primal LPP becomes. $\text{Max } z = 3x_1 - x_2 + x_3$

S. to $4x_1 - x_2 + 0x_3 \leq 8$

$-8x_1 - x_2 - 3x_3 \leq -12$

$5x_1 + 0x_2 - 6x_3 \leq 13$ & x_1, x_2, x_3 …

The dual LPP is $W = b^T y$ S. to $A^T y \geq C^T$

$\text{Min } W = 8y_1 - 12y_2 + 13y_3$ & $y \geq 0$.

S. to $4y_1 - 8y_2 + 5y_3 \geq 3$

$-y_1 - y_2 \geq -1$

$-3y_2 - 6y_3 \geq 1$ and $y_1, y_2, y_3 \geq 0$.

③ Construct the dual of the LPP.

$\text{Min } z = 4x_1 + 6x_2 + 18x_3$

S. to $x_1 + 3x_2 \geq 3$

$x_2 + 2x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$.

Soln: The given primal becomes

$\text{Min } z = 4x_1 + 6x_2 + 18x_3$

S. to $x_1 + 3x_2 + 0x_3 \geq 3$

$0x_1 + x_2 + 2x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$.

The dual LPP is

$\text{Max } W = 3y_1 + 5y_2$

S. to $y_1 \leq 4$

$3y_1 + y_2 \leq 6$

$2y_2 \leq 18$ and $y_1, y_2, y_3 \geq 0$.

④ Write the dual of the LPP

$\text{Min } z = 3x_1 - 2x_2 + 4x_3$

S. to $3x_1 + 5x_2 + 4x_3 \geq 7$

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 \leq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3$ … and x …

Soln. The given primal becomes.

Min $z = 3x_1 - 2x_2 + 4x_3$

S. to $3x_1 + 5x_2 + 4x_3 \geq 7$

$6x_1 + x_2 + 3x_3 \geq 4$

$-7x_1 + 2x_2 + x_3 \geq -10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$ and $x_1, x_2, x_3 \geq 0$.

The dual LPP is

Max $w = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

S. to $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

and $y_1, y_2, y_3, y_4, y_5 \geq 0$.

Unsymmetric Form:

	Primal / Dual	Dual / Primal
1.	Max $z = CX$ S. to $AX = b$ & $X \geq 0$.	Min $w = b^T y$ S. to $A^T y \geq C^T$ Variables are unrestricted
2.	Min $z = CX$ S. to $AX = b$ & $X \geq 0$	Max $w = b^T y$ S. to $A^T y \leq C^T$ Variables are unrestricted

1. Write the dual of the LPP Max $z = 3x_1 + 10x_2 + 2x_3$

Subject to $2x_1 + 3x_2 + 2x_3 \leq 7$

$3x_1 - 2x_2 + 4x_3 = 3$

and $x_1, x_2, x_3 \geq 0$.

Soln. Given primal LPP is

$\therefore$ The dual problem is Min $w = 7y_1 + 3y_2$

S. to $2y_1 + 3y_2 \geq 3$

$3y_1 - 2y_2 \geq 10$

$2y_1 + 4y_2 \geq 2$ and $y_1 \geq 0$, y_2 is unrestricted.

second eqn $= \text{so}$

2. Write the dual of the following primal LPP

Min $z = 4x_1 + 5x_2 - 3x_3$

S. to $x_1 + x_2 + x_3 = 22$

$3x_1 + 5x_2 - 2x_3 \leq 65$

$x_1 + 7x_2 + 4x_3 \geq 120$, $x_1 \geq 0$, $x_2 \geq 0$ and x_3 is unrestricted.

Soln. Given primal LPP is

Min $z = 4x_1 + 5x_2 - 3x_3$

S. to $x_1 + x_2 + x_3 = 22$

$-3x_1 - 5x_2 + 2x_3 \geq -65$

$x_1 + 7x_2 + 4x_3 \geq 120$

See back. Since the first constr. is eq.

The dual LPP is Max $w = 22y_1 - 65y_2 + 120y_3$

S. to $y_1 - 3y_2 + y_3 \leq 4$

$y_1 - 5y_2 + 7y_3 \leq 5$

$y_1 + 2y_2 + 4y_3 = -3$

and $y_2, y_3 \geq 0$, y_1 is unrestricted.

Since the first primal constraint is a ... the corresponding first dual variable y_1 ... unrestricted in sign. Also since the third variable x_3 is unrestricted in sign, the third dual constraint will be an equality

Problems based on Duality

1. Use duality to solve the following LPP

Minimize $Z = 2x_1 + 2x_2$

Subject to $2x_1 + 4x_2 \geq 1$

$-x_1 - 2x_2 \leq -1$

$2x_1 + x_2 \geq 1$ and $x_1, x_2 \geq 0$.

Soln: Given primal LPP is min $Z = 2x_1 + 2x_2$

min ≥ 1 S. to $2x_1 + 4x_2 \geq 1$

$x_1 + 2x_2 \geq 1$

$2x_1 + x_2 \geq 1$ and $x_1, x_2 \geq 0$.

Max

The dual LPP is Max $W = y_1 + y_2 + y_3$

S.to $2y_1 + y_2 + 2y_3 \leq 2$

$4y_1 + 2y_2 + y_3 \leq 2$ and $y_1, y_2, y_3 \geq 0$

By introducing the non-negative slack variables S_1 and S_2, the standard form of the dual becomes, Max $W = y_1 + y_2 + y_3 + 0S_1 + 0S_2$

Subject to $2y_1 + y_2 + 2y_3 + S_1 + 0S_2 = 2$

$4y_1 + 2y_2 + y_3 + 0S_1 + S_2 = 2$

and $y_1, y_2, y_3, S_1, S_2 \geq 0$.

<u>Third Iteration</u> : Introduce y_2 and drop y_1

C_B	y_B	x_B	y_1	y_2	y_3	S_1	S_2
	b_j		(1	1	1	0	0)
1	y_3	$2/3$	0	0	1	$2/3$	$-1/3$
1	y_2	$2/3$	2	1	0	$-1/3$	$2/3$
$(w_j - b_j)$		$4/3$	1	0	0	$1/3$	$1/3$
						x_1	x_2

Since all $(w_j - b_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution to the dual LPP is

Max $w = 4/3$, $y_1 = 0$, $y_2 = 2/3$, $y_3 = 2/3$.

Here it is observed that the primal variables x_1 and x_2 respectively correspond to the slack variables S_1 and S_2 of the dual problem.

As seen from above table, the net evaluations $(w_j - b_j)$ corresponding to the slack variables S_1 and S_2 are $1/3$ and $1/3$ respectively.

$\therefore$ The optimum solution to the original primal LPP is min $z = 4/3$, $x_1 = 1/3$ and $x_2 = 1/3$.

The initial basic feasible solution is given by

$R_1 = 3$, $R_2 = 2$ (basic) ($y_1 = y_2 = y_3 = y_4 = S_1 = S_2 = 0$ non-basic)

Initial Iteration

b_j ($1\ \ -7\ \ -10\ \ -3\ \ 0\ \ 0\ \ -M\ \ -M$)

C_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2	R_1	R_2	$\theta = \min \frac{x_B}{y_i}$
$-M$	R_1	3	-1	1	1	0	-1	0	1	0	3
$-M$	R_2	2	-1	1	(2)	1	0	-1	0	1	1
$(W_j^* - b_j)$		$-5M$	$2M-1$	$-2M+7$	$-3M+10$	$-M+3$	M	M	0	0	

First Iteration: Introduce y_3 and drop R_2.

b_j ($1\ \ -7\ \ -10\ \ -3\ \ 0\ \ 0\ \ -M$)

C_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2	R_1	θ
$-M$	R_1	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	4
-10	y_3	1	$-\frac{1}{2}$	$(\frac{1}{2})$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	2
$(W_j^* - b_j)$		$-2M-10$	$\frac{M+8}{2}$	$\frac{-M+4}{2}$	0	$\frac{M-4}{2}$	M	$\frac{-M+10}{2}$	0	

Second Iteration: Introduce y_2 and drop y_3.

b_j ($1\ \ -7\ \ -10\ \ -3\ \ 0\ \ 0\ \ -M$)

C_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2	R_1	θ
$-M$	R_1	1	0	0	-1	-1	-1	(1)	1	1
-7	y_2	2	-1	1	2	1	0	-1	0	—
$(W_j^* - b_j)$		$-M-14$ $-M-14$	6	0	$M-4$	$M-4$	M	$-M+7$	0	

Third Iteration: Introduce S_2 and drop R_1

b_j ($1\ \ -7\ \ -10\ \ -3\ \ 0\ \ 0$)

C_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2
0	S_2	1	0	0	-1	-1	-1	1
-7	y_2	3	-1	1	1	0	-1	0
$(W_j^* - b_j)$		-21	6	0	3	3	7	0

Since all $(W_j^* - b_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution of the dual problem is

$\text{Max } W^* = -21 \quad y_2 = 3 \quad y_1 = y_3 = y_4 = 0.$

i.e., $\text{Min } W = 21 \quad y_2 = 3,\ y_1 = y_3 = y_4 = 0.$

Also, from the above optimum simplex table of the dual problem, the optimum solution of the original (primal) problem is given by

$\text{Max } z = 21,\ x_1 = 7,\ x_2 = 0.$

Dual simplex method

1. Using dual simplex method solve the LPP

Minimize $Z = 2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$ and $x_1, x_2 \geq 0$.

Soln: The given LPP becomes

Max $z^* = -2x_1 - x_2$

Subject to $-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3$ and $x_1, x_2 \geq 0$.

By introducing the non-negative slack variables s_1, s_2 and s_3, the LPP becomes

Max $z^* = -2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to $-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_2 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$. ($s_1 = -3, s_2 = -6, s_3 = -3$ basic)

$x_1 = x_2 = 0$ (non-basic)

Initial Iteration

c_j (-2 -1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	-3	-3	-1	1	0	0
0	s_2	-6	-4	(-3)	0	1	0
0	s_3	-3	-1	-2	0	0	1
$(z_j^* - c_j)$		0	2	1	0	0	0

Since all $(z_j^* - c_j) \geq 0$ and all $x_{Bi} < 0$, the current solution is not an optimum basic feasible solution.

Since $x_{B2} = -6$ is the most negative, the corresponding basic variable s_2 leaves the basis.

Now $\theta = \text{Max}\left\{\frac{(z_j^* - c_j)}{a_{ik}}, a_{ik} < 0\right\}$ where x_k is the leaving variable.

$= \text{Max}\left\{\frac{2}{-4}, \frac{1}{-3}\right\} = \text{Max}\left\{-\frac{1}{2}, -\frac{1}{3}\right\} = -\frac{1}{3}$

$\therefore$ The corresponding non-basic variable x_2 enters into the basis.

First Iteration: Drop s_2 and introduce x_2.

c_j (-2 -1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	-1	$(-\frac{5}{3})$	0	1	$-\frac{1}{3}$	0
-1	x_2	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0
0	s_3	1	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1
$(z_j^* - c_j)$		-2	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0

Since all $(z_j^* - c_j) \geq 0$ and $x_{B1} = -1 < 0$, the current solution is not optimum basic feasible solution.

Since $x_{B1} = -1$ is negative, the corresponding basic variable s_1 leaves the basis.

Now $\theta = \text{Max}\left\{\frac{(z_j^* - c_j)}{a_{ik}}, a_{ik} < 0\right\}$

$= \text{Max}\left\{\frac{2/3}{-5/3}, \frac{1/3}{-1/3}\right\} = \text{Max}\left\{-\frac{2}{5}, -1\right\} = -\frac{2}{5}$

$\therefore$ The corresponding non-basic variable x_1 enters the basis.

S.to $-x_1 - x_2 \le -1$

$x_1 + x_2 \le 7$

$-x_1 - 2x_2 \le -10$

$0x_1 + x_2 \le 3$ and $x_1, x_2 \ge 0$.

By introducing the non-negative slack variables s_1, s_2, s_3 and s_4 the LPP becomes

Max $z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

Subject to $-x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$0x_1 + x_2 + s_4 = 3$ and $x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$.

The initial basic solution is given by

$s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3$ (basic)

($x_1 = x_2 = 0$ non-basic)

Initial Iteration

		c_j	(-3	-2	0	0	0	0)
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	-1	-1	-1	1	0	0	0
0	s_2	7	1	1	0	1	0	0
0	s_3	-10	-1	(-2)	0	0	1	0
0	s_4	3	0	1	0	0	0	1
$(z_j - c_j)$		0	3	2	0	0	0	0

First Iteration: Drop s_3 and introduce x_2.

Second Iteration: Drop s_1 and introduce x_1.

		c_j	(-2	-1	0	0	0)
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3
-2	x_1	3/5	1	0	-3/5	1/5	0
-1	x_2	6/5	0	1	4/5	-3/5	0
0	s_3	0	0	0	1	-1	1
$(z_j^* - c_j)$		$-\frac{12}{5}$	0	0	2/5	1/5	0

Since all $(z_j^* - c_j) \ge 0$ and all $x_{Bi} \ge 0$, the current solution is an optimum basic feasible solution.

$\therefore$ The optimum solution is Max $z^* = \frac{-12}{5}$

$x_1 = 3/5$, $x_2 = 6/5$

But min $z = -\text{Max } z^* = -\left(\frac{-12}{5}\right) = \frac{12}{5}$

$\therefore$ Min $z = \frac{12}{5}$, $x_1 = 3/5$, $x_2 = 6/5$.

2. Use dual simplex method to solve the LPP

Maximize $z = -3x_1 - 2x_2$

S.to $x_1 + x_2 \ge 1$

$x_1 + x_2 \le 7$

$x_1 + 2x_2 \ge 10$

$x_2 \le 3$ and $x_1, x_2 \ge 0$

Soln: The given LPP is Max $z = -3x_1 - 2x_2$

		C_j	(-3	-2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	4	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0
0	S_2	2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0
-2	x_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0
0	S_4	-2	$(-\frac{1}{2})$	0	0	0	$\frac{1}{2}$	1
$(Z_j - C_j)$		-10	2	0	0	0	1	0

<u>Second Iteration</u>: Drop S_4 and introduce x_1

		C_j	(-3	-2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	2	0	0	1	0	-1	-1
0	S_2	0	0	0	0	1	$\frac{1}{2}$	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
$(Z_j - C_j)$		-18	0	0	0	0	3	4

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is an optimum basic feasible solution.

$\therefore$ The optimum solution is

Max $Z = -18$, $x_1 = 4$, $x_2 = 3$.

$$\sum_{j=1} x_{ij} = a_i ,$$

$$\sum_{i=1}^{m} x_{ij} = b_j , \quad j = 1,2, \ldots n.$$

and $x_{ij} \geq 0$, for all i and j.

Types of TP:

(i) Balanced TP:

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then the Problem

(total supply) (total demand)

is said to be balanced TP.

(ii) unbalanced TP:

If the total supply is not equal to total demand then the Problem is said to be unbalanced TP.

i.e., $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$

* If $\sum a_i > \sum b_j$, then include a dummy destination in the Matrix with zero cost vectors. The excess supply is entered as a rim requirement ... dummy destination.

* If $\sum a_i < \sum b_j$, then include a dummy source in the ... with zero cost vectors. The excess ... and ... entered as a rim requirement ... the ... source.

A basic feasible [illegible] ... less than $m+n-1$ non-ve allocations is said to be a degenerate basic feasible solution.

Optimal Solution:

A feasible solution is said to be an optimal solution if it minimized the total transportation cost.

Methods for finding Initial Basic feasible Solution

The TP has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given TP is balanced.

Method 1: North West Corner Rule

(1) Determine basic feasible solution to the following TP using North-West Corner rule.

Sink

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	21

First check balance, total value is equal

Solution:

Since $a_i = b_j = 21$, the given Problem is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

Step 3 :

$\boxed{2}$ 4	7	2	1	~~8~~ 6
9	4	8	12	9
~~2~~	4	5	6	

Allocate $x_{22} = \min\{8, 2\} = 2$ to the cell (2,2) and move horizontally to the cell (2,3)

Step 4:

$\boxed{4}$ 7	2	1	~~6~~ 2
4	8	12	9
~~4~~	5	6	

Here $x_{23} = \min\{6, 4\} = 4$ and move horizontally to the cell (2,4)

Step 5 :

$\boxed{2}$ 2	1	~~2~~
8	12	9
~~5~~ 3	6	

Here $x_{24} = \min\{2, 5\} = 2$ and move vertically to the cell (3,4).

Step 6 :

$\boxed{3}$ 8	12	~~9~~ 6
~~3~~	6	

Here ... and move horizontally to the cell

Least Cost Method

① Find the initial basic feasible solution for the following transportation problem by least cost method.

	TO				Supply
From	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	100

Check: total is equal

Solution:

Since $\sum a_i = \sum b_j = 100$, the given TPP is balanced. ∴ There exist a feasible solution to the transportation problem.

Step 1: we've to allocate least value in this table

$\boxed{20}$ 1	2	1	4	~~30~~ 10
3	3	2	1	50
4	2	5	9	20
~~20~~	40	30	10	

By least cost method, $\min c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1,1) and allocate $x_{11} = \min\{a_1, b_1\} = \min\{30, 20\} = 20$ and cross out the satisfied column and decrease 30 by 20

The resulting reduced transportation table is

Step 2:

2	$\boxed{10}$ 1	4	~~10~~
3	2	1	50
2	5	9	20
40	~~30~~ 20	10	

Here $\min c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1,3) and allocate $x_{13} = \min\{10, 30\} = 10$ and cross out the satisfied row

The resulting reduced transportation table is

Step 3:

3	2	$\boxed{10}$ 1	~~50~~ 40
2	5	9	20
40	20	~~10~~	

Here $\min c_{ij} = c_{24} = 1$

∴ Allocate $x_{24} = \min\{50, 10\} = 10$ and cross out the satisfied column.

The resulting reduced transportation table is

Step 4:

3	$\boxed{20}$ 2	~~40~~ 20
2	5	20
40	~~20~~	

Here $\min c_{ij} = c_{23} = c_{32} = 2$

Choose the cell (2,3) and allocate $x_{23} = \min\{40, 20\} = 20$ and cross out the satisfied column. The resulting reduced transportation table is

Step 5:

	3	20
	[20] 2	~~20~~
	~~40~~ 20	

Here $\min c_{ij} = c_{32} = 2$ choose the cell (3,2) and allocate $x_{32} = \min\{40, 20\} = 20$ and cross out the satisfied row.

Step 6:

[20] 3	20
20	

Finally the initial basic feasible solution is as shown in the following table.

[20] 1	2	[10] 1	4
3	[20] 3	[20] 2	[10] 1
4	[20] 2	5	9

From this table we see that the number of positive independent allocations is equal to

Solution: Since $\sum a_i = \sum b_j = 950$, the given problem is balanced.

Step 1:

$\therefore$ There exists a feasible solution to this problem.

We're to take next-next smallest value in the table

$\boxed{200}$ 11	13	17	14	~~250~~ 50 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
~~200~~	225	275	250	
(5)	(5)	(1)	(0)	

We're to take largest value & take minimum element in that column.

First let us find the difference between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these differences is (5) and is associated with the first two columns of the transportation table. we choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate $x_{11} = \min\{250, 200\} = 200$ to this cell (1,1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

Step 2:

$\boxed{50}$ 13	17	14	~~50~~ (1)
18	14	10	300 (4)
24	13	10	400 (3)
~~225~~ 175	275	250	
(5)	(1)	(0)	

$11\times200 + 13\times50 + 18\times175 + 14\times125 + 13\times150 + 10\times250$

Step 3:

$\boxed{175}$ 18	14	10	~~300~~ 125 (4)
24	13	10	400 (3)
~~175~~	275	250	
(6)	(1)	(0)	

Step 4:

14	$\boxed{125}$ 10	~~300~~ 125 (4)
13	10	400 (3)
275	~~250~~ 125	
(1)	(0)	

$\boxed{125}$ 14	10	(4)
13	10	400 (3)
150 (1)	(0) 250	

Step 5:

13	$\boxed{125}$ 10	~~400~~ 275
275	~~125~~	

$\boxed{150}$ 13	10	~~400~~ 250
~~150~~	250	

Step 6:

$\boxed{275}$ 13	275
275	

Step 7: Finally the initial basic feasible Solution is as Shown in the following table.

$\boxed{200}$ 11	$\boxed{50}$ 13	17	14
16	$\boxed{175}$ 18	14	$\boxed{125}$ 10
21	24	$\boxed{275}$ 13	$\boxed{125}$ 10

$m+n-1$
$3+4-1$
$7-1=6$

From this table we see that the number of Positive independent allocations is equal to $m+n-1 = 3+4-1 = 6$. This ensures that the Solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost is

$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$=$ Rs 12075 /-

②

1	2	6	7	(1)
0	4	$\boxed{10}$ 2	$\cancel{12}$ 2	(2)
3	1	5	11	(2)
10	10	$\cancel{16}$		
(1)	(1)	(3)		

Transportation Algorithm (or) MODI Method (Modified distribution method)

① Solve the transportation Problem

V.Q

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

balanced

Solution:

Since $\sum a_i = \sum b_j = 43$, the given transportation problem is balanced.

∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table.

21	16	25	$\boxed{11}$ 13	~~11~~	(3)		
$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	~~7~~ 3, ~~13~~	(3)	(3)	(1)
32	$\boxed{7}$ 27	$\boxed{12}$ 18	41	7, ~~19~~	(9)	(9)	(5
~~6~~	~~10~~ 7	~~12~~	~~15~~ 4				
(4)	(2)	(4)	(10)				
(5)	(9)	(4)	(18)				

m+n-1

3+4-1

7-1

= 6

$21-7=14$ $c_{11}=v_1+u_1=$ ...

$d_{11}=c_{11}-(u_1+v_1)$

	[14] 21	[7] 8 16	[8] 26 25	[11] 13	$u_1=-10$
	[6] 17	[3] 18	[5] [9] 4 14	23	$u_2=0$
	[6] 32	[26] 7 27	[12] 18	[9] 41 [32]	$u_3=9$
	$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$	

For unoccupied cells:

$$d_{ij}=c_{ij}-(u_i+v_j)$$

$$d_{11}=c_{11}-(u_1+v_1)=21-7=14$$

$d_{12}=8$, $d_{13}=26$, $d_{23}=5$, $d_{31}=6$, $d_{34}=9$

Since all $d_{ij}>0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{14}=11$, $x_{21}=6$, $x_{22}=3$, $x_{24}=4$, $x_{32}=7$, $x_{33}=12$

$\therefore$ The minimum transportation cost is

$= 13\times 11 + 17\times 6$ ... $\times 3 + 23\times 4 + 27\times 7 + 18\times 12$

$=$ ...

(2) Find the optimal transportation cost of ... Matrix using least cost method for finding the critical solution.

		Market					
		A	B	C	D	E	Available
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
	Demand	40	50	70	90	90	340 balanced

Solution: Since $\sum a_i=\sum b_j=340$, the given T.P is balanced.

$\therefore$ There exist a basic feasible solution to this problem. By using LCM, the initial solution is shown in the following table.

4	[50] 1	[50] 2	6	9	~~100~~ ~~50~~
[10] 6	4	[20] 3	5	[90] 7	~~120~~ ~~100~~ 90
[30] 5	2	6	[90] 4	8	~~120~~ ~~30~~
~~40~~ 10	~~50~~	~~70~~ 20	~~90~~	90	

$\therefore$ The initial transportation cost is

$= 1\times 50 + 2\times 50 + 6\times 10 + 3\times 20 + 7\times 90 + 5\times 30 + 4\times 90$

$=$ Rs. 1410.

Scanned by CamScanner

For optimality: Since the number of non-ve independent allocations is $m+n-1$, we apply MODI m...

$d_{11} = c_{11} - (u_1+v_1) = 4-5 = -1$ $\quad c_{11} = u_1+v_1 = -1+6 = 5$

4 [-1]	1 [50]	2 [50]	6 [2]	9 [3]	$u_1 = -1$
6 [10]	4 [2]	3 [20]	5 [0]	7 [90]	$u_2 = 0$ maximum allocation
5 [30]	2 [1]	6 [4]	4 [90]	8 [2]	$u_3 = -1$

$v_1 = 6$, $v_2 = 2$, $v_3 = 3$, $v_4 = 5$, $v_5 = 7$

$c_{ij} = u_i + v_j$ (for unoccupied cells)

~~$c_{22} = u_2+v_2 \Rightarrow 4 = 0+v_2 \Rightarrow v_2 = 4$~~

~~$c_{11} = u_1+v_1 \Rightarrow 4 =$~~

~~$c_{14} = u_1+v_4 \Rightarrow 6 =$~~

allocation cells

$c_{21} = u_2+v_1 \Rightarrow 6 = 0+v_1 \Rightarrow v_1 = 6$

$c_{23} = u_2+v_3 \Rightarrow 3 = 0+v_3 \Rightarrow v_3 = 3$

$c_{25} = u_2+v_5 \Rightarrow 7 = 0+v_5 \Rightarrow v_5 = 7$

$c_{31} = u_3+v_1 \Rightarrow 5 = u_3+6 \Rightarrow u_3 = -1$

$c_{34} = u_3+v_4 \Rightarrow 4 = -1+v_4 \Rightarrow v_4 = 5$

$c_{12} = u_1+v_2 \Rightarrow 1 = -1+v_2 \Rightarrow v_2 = 2$

$c_{13} = v_3+u_1 \Rightarrow 2 = 3+u_1 \Rightarrow u_1 = -1$

To find $d_{ij} = -(u_i+v_j) + c_{ij}$

Since $d_{11} = -1 < 0$, the current solution is not optimal.

-1 is -ve value so we've to take θ.

4 $+\theta$	1 [50]	2 [50] $-\theta$	6	9
6 [10] $-\theta$	4	3 [20] $+\theta$	5	7 [90]
5 [30]	2	6	4 [90]	8

From the two cells (1,3), (2,1) having $-\theta$, we find that the minimum of the allocations 50, 10 is 10.

Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence the new basic feasible solution is displayed in the following table

4 [10]	1 [50]	2 [40]	6	9
6	4	3 [30]	5	7 [90]
5 [30]	2	6	4 [90]	8

We see that the above table satisfies the rim conditions with $(m+n-1)$ non-negative allocations at independent positions. So we apply MODI method

Scanned by CamScanner

[10] 4	[50] 1	[40] 2	[3] 6 3	[3] 9 6	$u_1=0$
[1] 6 5	[2] 4 2	[30] 3	[1] 5 4	[90] 7	$u_2=1$
[30] 5	[0] 2 2	[3] 6 3	[90] 4	[1] 8 7	$u_3=1$
$v_1=4$	$v_2=1$	$v_3=2$	$v_4=3$	$v_5=6$	

$c_{11} = u_1+v_1 \Rightarrow 4 = 0+v_1 \Rightarrow v_1 = 4$

$c_{12} = u_1+v_2 \Rightarrow 1 = 0+v_2 \Rightarrow v_2 = 1$

$c_{13} = u_1+v_3 \Rightarrow 2 = 0+v_3 \Rightarrow v_3 = 2$

$c_{23} = u_2+v_3 \Rightarrow 3 = u_2+2 \Rightarrow u_2 = 1$

$c_{25} = u_2+v_5 \Rightarrow 7 = 1+v_5 \Rightarrow v_5 = 6$

$c_{31} = u_3+v_1 \Rightarrow 5 = u_3+4 \Rightarrow u_3 = 1$

$c_{34} = u_3+v_4 \Rightarrow 4 = 1+v_4 \Rightarrow v_4 = 3$

Since $d_{ij} = c_{ij} - (u_i+v_j) > 0$, with $d_{32} = 0$, the current solution is optimal and there exist an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given

$x_{11} = 10$, $x_{12} = 50$, $x_{13} = 40$, $x_{23} = 30$, $x_{25} = 90$

$x_{31} = 30$ $x_{34} = 90$.

$\therefore$ The optimum transportation cost is

$= 4\times10 + 1\times50 + 2\times40 + 3\times30 + 7\times90 + 5\times30 + 4\times90$

= Rs. 1400 /-

Degeneracy

Degeneracy in Transportation Problems:

In a TP, whenever the number of non-ve independent allocations is less than m+n-1, the TP is said to be a degenerate one.

(1) Find the non-degenerate basic feasible soln for the transportation problem

	To				Supply
From	10	20	5	7	10
	13	9	12	8	20
	4	5	7	9	30
	14	7	1	0	40
	3	12	5	19	50
Demand	60	60	20	10	

This problem is using 3 methods
1. LCM
2. NWC
3. Vogel's method

We've to solve NWC

Soln. Since $\sum a_i = \sum b_j = 150$, the given TP is balanced.

Since the number of non-ve allocations at independent positions, is 7 which is less than $(m+n-1) = 5+4-1 = 8$, the basic feasible solution is degenerate ~~one~~ one.

To resolve this degeneracy, we allocate a very small quantity ϵ to the unoccupied cell (5,1) so that the number of occupied cells becomes m+n-1.

NWC

[10] 10	20	5	7	~~10~~
[20] 13	9	12	8	~~20~~
[30] 4	5	7	9	30
14	[40]	1	0	~~40~~
[ϵ] 3	[20] 12	[20] 5	[10] 19	~~50~~ ~~30~~ 10
~~60~~ ~~50~~ 30	60 ~~20~~	20	10	

Min allocation in NWC.

degeneracy

$m+n-1 < 0$

The initial transportation cost is

$= 10\times10 + 13\times20 + 4\times30 + 7\times40 + 3\times\epsilon + 12\times2$

$+ 5\times20 + 19\times10$

$= 1290 + 3\epsilon$

$=$ Rs $1290/-$ as $\epsilon \to 0$.

Unbalanced Transportation Problems:

(1) Solve the TP

Source \ Destination	A	B	C	D	Supply
1	11	20	7	8	50
2	21	16	20	12	40
3	8	12	18	9	70
Demand :	30	25	35	40	

Soln: Since the total supply $\sum a_i = 160$ is greater than the total demand $\sum b_j = 130$, the given problem is an unbalanced TP.

To convert this into a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to $160 - 130 = 30$ units.

$\therefore$ The given problem becomes

Source \ Destination	A	B	C	D	E	Supply
1	11	20	7	8	0	50
2	21	16	20	12	0	40
3	8	12	18	9	0	70
Demand :	30	25	35	40	30	

$C_{13} = u_1 + v_3 \Rightarrow 7 = -1 + v_3 \Rightarrow v_3 = 8$

$C_{31} = v_1 + u_3 \Rightarrow 8 = v_1 + 0 \Rightarrow v_1 = 8$

$C_{32} = v_2 + u_3 \Rightarrow 12 = v_2 + 0 \Rightarrow v_2 = 12$

$C_{34} = v_4 + u_3 \Rightarrow 9 = v_4 + 0 \Rightarrow v_4 = 9$

$C_{24} = v_4 + u_2 \Rightarrow 12 = 9 + u_2 \Rightarrow u_2 = 3$

$C_{25} = v_5 + u_2 \Rightarrow 0 = v_5 + 3 \Rightarrow v_5 = -3$

$C_{14} = v_4 + u_1 \Rightarrow 8 = 9 + u_1 \Rightarrow u_1 = -1$

Since all $d_{ij} > 0$, the solution under the (e)
is optimal and unique.

$\therefore$ The optimum allocation schedule is

$x_{13} = 35$, $x_{14} = 15$, $x_{24} = 10$, $x_{25} = 30$,

$x_{31} = 30$, $x_{32} = 25$, $x_{34} = 15$

$\therefore$ The minimum transportation cost

$= Rs\ 7 \times 35 + 8 \times 15 + 12 \times 10 + 0 \times 30 +$
$8 \times 30 + 12 \times 25 + 9 \times 15$

$= Rs\ 1160/-$

(1) Consider the problem of assigning five jobs ~~and~~ to five persons. The assignment costs are given as follows.

$$
\begin{array}{cc}
 & \text{Job} \\
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\text{Person} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{bmatrix}
8 & 4 & 2 & 6 & 1 \\
0 & 9 & 5 & 5 & 4 \\
3 & 8 & 9 & 2 & 6 \\
4 & 3 & 1 & 0 & 3 \\
9 & 5 & 8 & 9 & 5
\end{bmatrix}
\end{array}
$$

Determine the optimum assignment schedule.

Solution:

Since the number of rows is equal to number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: Select smallest cost element in each row and subtract this from all the elements of corresponding row, we get.

$$
\begin{bmatrix}
7 & 3 & 1 & 5 & 0 \\
0 & 9 & 5 & 5 & 4 \\
1 & 6 & 7 & 0 & 4 \\
4 & 3 & 1 & 0 & 3 \\
4 & 0 & 3 & 4 & 0
\end{bmatrix}
$$

Step 2: Select smallest cost element in each column and subtract this from all the elements of corresponding column, we get reduced matrix.

$$
\begin{bmatrix}
7 & 3 & 0 & 5 & 0 \\
0 & 9 & 4 & 5 & 4 \\
1 & 6 & 6 & 0 & 4 \\
4 & 3 & 0 & 0 & 3 \\
4 & 0 & 2 & 4 & 0
\end{bmatrix}
$$

Step 3: Examine the rows successively until a row with exactly one unmarked zero is found.

$A \to 5$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{bmatrix}
7 & 3 & \cancel{0} & 5 & (0) \\
(0) & 9 & 4 & 5 & 4 \\
1 & 6 & 6 & (0) & 4 \\
4 & 3 & (0) & \cancel{0} & 3 \\
4 & (0) & 2 & 4 & \cancel{0}
\end{bmatrix}
\end{array}
$$

Each one row atleast allocation of zero exist. Then stop the process.

Step 4: Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

$A \to 5$, $B \to 1$, $C \to 4$, $D \to 3$, $E \to 2$

The optimum assignment cost $= \underline{1+0+2+1+5}$ (from given matrix)

$= 9$ unit of cost.

(2) The Processing time in hours for the jobs when allocated to the different Machines are indicated below. Assign the Machines for the jobs so that the total processing time is minimum

Machines

Jobs	M_1	M_2	M_3	M_4	M_5
J_1	9	22	58	11	19
J_2	43	78	72	50	63
J_3	41	28	91	37	45
J_4	74	42	27	49	39
J_5	36	11	57	22	25

Solution:

Since No of rows = No of columns, the given assignment problem is balanced.

Step 1:

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Step 6: Here 4 is the smallest element not covered by these straight lines.

Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of these st. lines.

$$
\begin{array}{c}
\begin{array}{ccccc} & \text{Add } 4 & & & \end{array} \\
\begin{bmatrix}
0 & \underline{17} & 49 & 0 & 0 \\
0 & \underline{39} & 29 & 5 & 10 \\
9 & 0 & 59 & 3 & 3 \\
47 & \underline{19} & 0 & 20 & 2 \\
21 & 0 & 42 & 5 & 0
\end{bmatrix}
\end{array}
$$

This row subtract by

This row subtract by

Since each row and each column contains at least one zero, we examine the rows successively.

$$
\begin{array}{c|ccccc}
 & M_1 & M_2 & M_3 & M_4 & M_5 \\
\hline
J_1 & \cancel{0} & 17 & 49 & {}^{5th}(0) & \cancel{0} \\
J_2 & {}^{1st}(0) & 39 & 29 & 5 & 10 \\
J_3 & 9 & {}^{2nd}(0) & 59 & 3 & 3 \\
J_4 & 47 & 19 & {}^{3rd}(0) & 20 & 2 \\
J_5 & 21 & \cancel{0} & 42 & 5 & {}^{4th}(0)
\end{array}
$$

In the above matrix, each row and each column contains exactly one assignment, therefore the current solution is optimal.

nbalanced Assignment Models:

No of rows ≠ No of columns, the given assignment problem is unbalanced.

First convert unbalanced assignment problem into a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

(1) A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Jobs \ Machines	1	2	3	4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

What are job assignments which will minimize the cost?

Soln:

No of rows < No of columns, the given problem is unbalanced.

Step 1: To make it a balanced one, add a dummy row with zero cost elements.