Initial Iteration:

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
	c_j		1	1	0	0	
0	x_3	5	(3)	2	1	0	$5/3$ *
0	x_4	2	0	1	0	1	—
	$(z_j - c_j)$	0	-1	-1	0	0	

Since some $(z_j - c_j) < 0$, the current basic feasible solution is not optimal.

First Iteration: Introduce x_1 and drop x_3.

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
	c_j		1	1	0	0	
1	x_1	$5/3$	1	$2/3$	$1/3$	0	$5/2$
0	x_4	2	0	(1)	0	1	2 *
	$(z_j - c_j)$	$5/3$	0	$-1/3$	$1/3$	0	

Since some $(z_j - c_j) < 0$, the current basic feasible solution is not optimal.

Second Iteration: Introduce x_2 and drop x_4

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
	c_j		1	1	0	0
1	x_1	$1/3$	1	0	$1/3$	$-2/3$
1	x_2	2	0	1	0	1
	$z_j - c_j$	$7/3$	0	0	$1/3$	$1/3$

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal and non-integer.

$$\text{Max } z = 7/3, \quad x_1 = 1/3, \quad x_2 = 2.$$

To obtain the optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_1 = 1/3$, from the source row (first row)

$x_1 \quad 1/3 \quad 1 \quad 0 \quad 1/3 \quad -2/3$

we have $1/3 = x_1 + 1/3\, x_3 - 2/3\, x_4$

Expressing the negative fraction as a sum of a negative integer and non-negative fraction, we have

$$1/3 = x_1 + 1/3\, x_3 + (-1 + 1/3) x_4$$

The fractional cut (Gomorian) constraint is given by

$$1/3\, x_3 + 1/3\, x_4 \geq 1/3$$

$$\Rightarrow -1/3\, x_3 - 1/3\, x_4 \leq -1/3$$

$$\Rightarrow -1/3\, x_3 - 1/3\, x_4 + S_1 = -1/3$$

where S_1 is the Gomorian slack.

Add this fractional cut constraints at the bottom of the above optimum simplex table, we have the new simplex table.

C_j (1 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	0
1	x_2	2	0	1	0	1	0
0	S_1	$-1/3$	0	0	$(-1/3)$	$-1/3$	1
	$Z_j - C_j$	$7/3$	0	0	$1/3$	$1/3$	0

old eqn -

Here the solution is optimal but infeasible.

To obtain the feasible optimal solution, we have to use the dual simplex method.

Since $S_1 = -1/3$, S_1 leaves the basis.

To find the entering variable let $\max\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\}$

$= \max\left\{\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right\} = \max\{-1, -1\} = -1$ which corresponds to both x_3 and x_4. We choose x_3 as the entering variable arbitrarily.

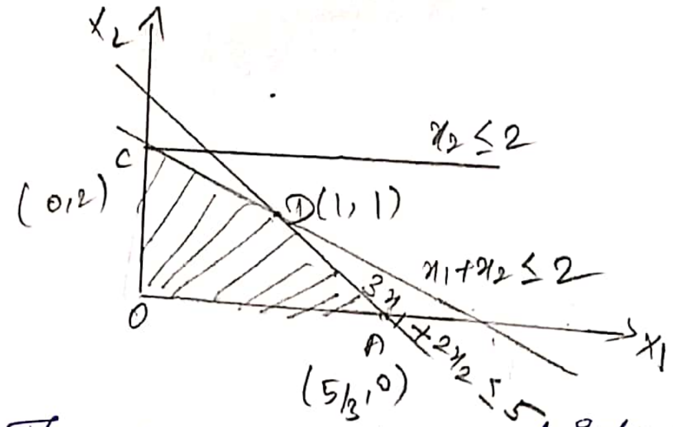
<u>Third</u> <u>Iteration</u>: Drop S_1 and introduce x_3.

C_j (1 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	x_3	1	0	0	1	1	-3
	$(Z_j - C_j)$	2	0	0	0	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal and integer.

Drawing the line $x_1+x_2=2$, the above feasible region is cut off to the shaded region shown below.



Thus the required optimal integer valued solution is Max $z=2$, $x_1=0$, $x_2=2$ (or)

Max $z=2$, $x_1=1$, $x_2=1$

2. Solve the following LPP Min $z=-2x_1-3x_2$

Subject to constraints $2x_1+2x_2 \le 7$

$x_1 \le 2$

$x_2 \le 2$, $x_1, x_2 \ge 0$ and integers.

Soln: Given L.P.P is Min $z=-2x_1-3x_2$

Subject to $2x_1+2x_2 \le 7$

$x_1 \le 2$, $x_2 \le 2$, $x_1, x_2 \ge 0$ and integ

i. Maximize $z^* = 2x_1+3x_2$

Subject to $2x_1+2x_2 \le 7$

$x_1 \le 2$, $x_2 \le 2$, $x_1, x_2 \ge 0$ and integ

The optimum integer solution ...

Second Iteration: Introduce x_1 and drop x_3.

		c_j	2	3	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
2	x_1	$3/2$	1	0	$1/2$	0	-1
0	x_4	$1/2$	0	0	$-1/2$	1	1
3	x_2	2	0	1	0	0	1
$Z_j - C_j$		9	0	0	1	0	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal. but non-integer.

To obtain the optimum integer solution, we have to construct a fractional cut constant.

Now, $x_1 = 3/2 = 1 + 1/2 = [X_{B_1}] + f_1$

$x_4 = 1/2 = 0 + 1/2 = [X_{B_2}] + f_2$

$\therefore \text{Max}\{f_1, f_2\} = \text{Max}\{1/2, 1/2\} = 1/2$ which

corresponds to both first and second rows.

We select first row arbitrarily as the source row.

$3/2 = x_1 + 1/2\, x_3 - x_5$

$1 + 1/2 = x_1 + 1/2\, x_3 - x_5$

$\therefore$ The fractional cut (Gomorian) constraint is given by $1/2\, x_3 \geq 1/2 \Rightarrow -1/2\, x_3 \leq -1/2$

$\Rightarrow -1/2\, x_3 + S_1 = -1/2$ where S_1 is the Gomorian

∴ The optimal integer solution is

$Max\ z^* = 8,\ x_1 = 1,\ x_2 = 2$.

But $Min\ z = -Max(-z) = -Max\ z^* = -8$.

∴ $Min\ z = -8,\ x_1 = 1,\ x_2 = 2$.

Gomory's Mixed Integer method

1. Solve the following Mixed integer Programming Problem $Max\ z = x_1 + x_2$ Subject to Constraints

$2x_1 + 5x_2 \le 16$, $6x_1 + 5x_2 \le 30$, $x_2 \ge 0$

x_1 - non-negative integer

Soln: Introduce the non-negative slack variables x_3 and x_4, the standard form of the continuous LPP becomes $Max\ z = x_1 + x_2 + 0x_3 + 0x_4$

Subject to $2x_1 + 5x_2 + x_3 + 0x_4 = 16$

$6x_1 + 5x_2 + 0x_3 + x_4 = 30$

and $x_1, x_2, x_3, x_4 \ge 0$

The initial basic feasible solution is given by $x_3 = 16$, $x_4 = 30$, ($x_1 = x_2 = 0$, non-basic)

Initial Iteration:

		c_j	1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	16	2	5	1	0	8
0	x_4	30	(6)	5	0	1	5
$(z_j - c_j)$		0	-1	-1	0	0	

		c_j	1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	6	0	$(10/3)$	1	$-1/2$	$9/5$
1	x_1	5	1	$5/6$	0	$1/6$	6
$(z_j - c_j)$		5	0	$-1/6$	0	$1/6$	

<u>Second Iteration</u> Introduce x_2 and drop x_3

		c_j	1	1	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_2	$\frac{18}{10}$	0	1	$\frac{3}{10}$	$-1/10$
1	x_1	$7/2$	1	0	$-1/4$	$1/4$
$z_j - c_j$		$\frac{53}{10}$	0	0	$1/20$	$3/20$

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal. Since the integer constrained variable x_1 is non-integer, we have from the second (source) row $7/2 = x_1 + 0x_2 - 1/4\, x_3 + 1/4\, x_4$

$$3 + 1/2 = x_1 + 0x_2 - 1/4\, x_3 + 1/4\, x_4$$

The Gomorian constraint is given by

$$\left(\frac{1}{2}\right)\left(-1/4\right)x_3 + 1/4\, x_4 \geq 1/2$$

Branch and Bound Method

1. Use Branch and Bound Method to solve the following Maximize $Z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$, $x_1 + 2x_2 \leq 4$ and

$x_1, x_2 \geq 0$ and integer.

Soln.: Ignoring the integrality condition and introducing the non-negative slack variables x_3, x_4, the standard form of the continuous LPP becomes

Maximize $Z = 2x_1 + 2x_2 + 0x_3 + 0x_4$

Subject to $5x_1 + 3x_2 + x_3 + 0x_4 = 8$

$x_1 + 2x_2 + 0x_3 + x_4 = 4$

and $x_1, x_2, x_3, x_4 \geq 0$.

The initial basic feasible solution is given by

$x_3 = 8$, $x_4 = 4$ (basic) ($x_1 = x_2 = 0$, non-basic)

Initial Iteration:

	c_j		2	2	0	0	
C_B	y_B	x_B	x_1	x_2	x_3	x_4	θ
0	x_3	8	(5)	3	1	0	8/5
0	x_4	4	1	2	0	1	4
$Z_j - C_j$		0	-2	-2	0	0	

First Iteration: Introduce x_1 and drop x_3.

c_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
2	x_1	$8/5$	1	$3/5$	$1/5$	0	$8/3$
0	x_4	$12/5$	0	$(7/5)$	$-1/5$	1	$12/7$
	$z_j - c_j$	$16/5$	0	$-4/5$	$2/5$	0	

Second Iteration: Introduce x_2 and drop x_4

c_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$4/7$	1	0	$2/7$	$-3/7$
2	x_2	$12/7$	0	1	$-1/7$	$5/7$
	$z_j - c_j$	$\frac{32}{7}$	0	0	$2/7$	$4/7$

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal, but non-integer.

$Max\ z = \frac{32}{7}$, $x_1 = 4/7$, $x_2 = 12/7$

In order to obtain the integer optimal solution, we have to branch this problem into two sub-problem

Now $x_2 = \frac{12}{7} \Rightarrow 1 < x_2 < 2$

$x_2 \leq 1$ or $x_2 \geq 2$

Applying these two conditions separately in the continuous LPP, we have two sub-problems.

Sub Problem (1)

$Max\ z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_2 \leq 1$ and $x_1, x_2 \geq 0$

~~Its optimal solution is Max z = 4~~

Introduce the slack variables x_3, x_4, x_5 we get.

$Max\ z = 2x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $5x_1 + 3x_2 + x_3 + 0x_4 + 0x_5 = 8$

$x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 = 4$

$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 1$

Here $x_3 = 8$, $x_4 = 4$, $x_5 = 1$ (basic) $x_1 = x_2 = 0$ (non-basic)

Initial Iteration:

c_j (2 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
0	x_3	8	(5)	3	1	0	0	$8/5$ * $= 1.6$
0	x_4	4	1	2	0	1	0	4
0	x_5	1	0	1	0	0	1	0
	$z_j - c_j$	0	-2	-2	0	0	0	

First Iteration: Introduce x_1 and drop x_3

		c_j	2	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
2	x_1	$8/5$	1	$3/5$	$1/5$	0	0	$8/3$
0	x_4	$12/5$	0	$7/5$	$-1/5$	1	0	$12/7$
0	x_5	1	0	(1)	0	0	1	1 ←
	$z_j - c_j$	$16/5$	0	$-4/5$	$2/5$	0	0	

Second Iteration: Introduce x_2 and drop x_5

		c_j	2	2	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
2	x_1	1	1	0	0	0	$31/15$
0	x_4	1	0	0	$-1/5$	1	$-7/15$
2	x_2	1	0	1	0	0	1
	$z_j - c_j$	4	0	0	0	2	$\frac{32}{15}$

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is $\max z = 4$,

$x_1 = 1, \; x_2 = 1$.

Sub Problem (2)

$\text{Max } z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_2 \geq 2$ and $x_1, x_2 \geq 0$.

Introduce the slack variables x_3, x_4, x_5 and surplus variables also.

$\text{Max } z = 2x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5 - MR_1$

$5x_1 + 3x_2 + x_3 + 0x_4 + 0x_5 = 8$

$x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 = 4$

$0x_1 + x_2 + 0x_3 + 0x_4 - x_5 + R_1 = 2$

Here $x_3 = 8$, $x_4 = 4$, $R_1 = 2$ (basic) ($x_2 = x_1 = x_5 = 0$ non-basic)

Initial ~~First~~ Iteration:

		c_j	2	2	0	0	0	$-M$	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	R_1	θ
0	x_3	8	5	3	1	0	0	0	$8/3$
0	x_4	4	1	(2)	0	1	0	0	2 ←
$-M$	R_1	2	0	1	0	0	-1	1	2
	$z_j - c_j$	$-2M$	-2	$-M-2$ ✓	0	0	M	0	

First Iteration: Introduce x_2 and drop x_4.

		c_j	2	2	0	0	0	$-M$
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	R_1
0	x_3	2	$7/2$	0	1	$-3/2$	0	0
2	x_2	2	$1/2$	1	0	$1/2$	0	0
$-M$	R_1	0	$-1/2$	0	0	$-1/2$	-1	1
	$z_j - c_j$	4	$\frac{M}{2}-1$	0	0	$1+\frac{M}{2}$	M	0

Since all $(z_j - c_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 4$,

$x_1 = 0$, $x_2 = 2$.

So, this sub-problem is also fathomed.

Hence from both the sub-problem ① and ②, the integer optimum solution is given by

Max $z = 4$

with $x_1 = 1$, $x_2 = 1$ or $x_1 = 0$, $x_2 = 2$

Original Problem

Max $z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$ and $x_1, x_2 \geq 0$

Max $z = \frac{32}{7}$, $x_1 = 4/7$, $x_2 = 12/7$.

$x_2 \leq 1$ | $x_2 \geq 2$

Sub problem (1)

Max $z = 4$

$x_1 = 1$, $x_2 = 1$

Fathomed

Subproblem (2)

Max $z = 4$

$x_1 = 0$, $x_2 = 2$

Fathomed

Hence the integer optimum solution is

Max $z = 4$

with $x_1 = 1$, $x_2 = 1$ (or) $x_1 = 0$, $x_2 = 2$.

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DYNAMIC PROGRAMMING

Optimal Subdivision Problems

① Divide a positive quantity c in n-parts in such a way that their product is maximum (or) Maximize $Z = y_1 y_2 \cdots y_n$ subject to the constraints

$$y_1 + y_2 + \cdots + y_n = c \text{ and } y_i \geq 0,\ i = 1, 2, \cdots n.$$

Solution:

First we shall develop a recursive eqn connecting the optimal decision function for the n-stage problem with the optimal decision function for the $(n-1)$ stage sub-problem.

Let y_i be the ith part of c and each i may be regarded as a state. Since y_i may assume any non-negative value which satisfies $y_1 + y_2 + \cdots + y_n = c$, the alternative at each stage are infinite. This means y_i is continuous. Hence the optimal decisions at each stage are obtained by using usual classical method (differentiation).

Let $f_n(c)$ be the maximum attainable product $y_1 \cdot y_2 \cdot y_3 \cdots y_n$ when c is divided into n parts $y_1, y_2, \cdots y_n$. Thus $f_n(c)$ becomes a function of n. $f_n(c) = y_1, y_2 \cdots y_n$.

Stage 1: For $n=1$ (one stage problem):

If c is divided into one part only, they $y_1 = c$

$n = 1$,

$\therefore f_n(c) = c$ (trival case) $\longrightarrow$ (1)

$f_1(c) = c$.

For $n=2$ (Two stage Problem)

Here c is divided into two parts $y_1 = x$ and $y_2 = c - x$ such that $y_1 + y_2 = c$. Then

$y_1 + y_2 + \dots + y_n = c$

$$f_2(c) = \max_{0 \le x \le c} \{y_1 \cdot y_2\} = \max_{0 \le x \le c} \{x(c-x)\}$$

$x + c - x = c$

$c = c$.

$$= \max_{0 \le x \le c} \{x \, f_1(c-x)\} \qquad \because f_1(c) = c \rightarrow (2)$$

For $n=3$ (Three stage Problem)

Here c is divided into three parts y_1, y_2, y_3.

Let $y_1 = x$ and $y_2 + y_3 = c - x$ so that $y_1 + y_2 + y_3 = c$

i.e., $(c-x)$ is further divided into two parts whose maximum attainable product $y_2 y_3$ is $f_2(c-x)$

$$\text{Then } f_3(c) = \max_{0 \le x \le c} \{y_1 \cdot y_2 \cdot y_3\} = \max_{0 \le x \le c} \{x \, f_2(c-x)\}$$

In general, the recursive equation for the n-stage problem is $f_n(c) = \max_{0 \le x \le c} \{x \, f_{n-1}(c-x)\} \rightarrow (3)$

To solve the recursive equation:

For $n=2$, eqn (3) becomes $f_2(c) = \max_{0 \le x \le c} \{x \, f_1(c-x)\}$

$$f_2(c) = \max_{0 \le x \le c} \{x(c-x)\}$$

The function $x(c-x)$ will the maximum if

$$\frac{d}{dx}\left(x(c-x)\right) = 0 \Rightarrow x(-1) + (c-x) = 0$$

$$\Rightarrow -x + c - x = 0 \Rightarrow c - 2x = 0 \Rightarrow \boxed{x = c/2}$$

$$\therefore f_2(c) = \max_{0 \le x \le c} \{x(c-x)\} = c/2\,(c - c/2)$$

$$f_2(c) = (c/2)^2$$

The optimal policy is $(\frac{c}{2}, c/2)$ and $f_2(c) = (c/2)^2$

$f_2(c-x) = \left(\frac{c-x}{2}\right)^2$

For $n=3$, eqn (3) becomes

$$f_3(c) = \max_{0 \le x \le c} \{x \, f_2(c-x)\} = \max_{0 \le x \le c} \left\{x\left(\frac{c-x}{2}\right)^2\right\}$$

The function $x\left(\frac{c-x}{2}\right)^2$ will attain its maximum value at $x = (c/3)$

$$\frac{d}{dx}\left[x\left(\frac{c-x}{2}\right)^2\right] = 0$$

$$x\,2(c-x)(-1) + (c-x)^2 = 0$$

$$-2x(c-x) + (c-x)^2 = 0$$

$$[c-x][-2x + c - x] = 0$$

$$-3x + c = 0 \Rightarrow 3x = c$$

$$\boxed{x = c/3}$$

$$\therefore f_3(c) = \max_{0 \le x \le c} \left\{x\left(\frac{c-x}{2}\right)^2\right\} = \frac{c}{3}\left(\frac{c - c/3}{2}\right)^2 = (c/3)^3$$

The optimal policy is $(c/3, c/3, c/3)$ and $f_3(c) = (c/3)^3$

Let us assume that the optimal policy for $n=m$ is $(c/m, c/m, \dots c/m)$ and $f_m(c) = (c/m)^m$

Now for $n = m+1$, eqn (3) becomes

$$f_{m+1}(c) = \max_{0 \le x \le c} \{x \, f_m(c-x)\} = \max_{0 \le x \le c} \left\{x\left(\frac{c-x}{m}\right)^m\right\};$$

The function $x\left(\frac{c-x}{m}\right)^m$ attains its maximum value at $x = c/_{m+1}$

$$\frac{d}{dx}\left[x(c-x)^m\right] = 0$$

$$x\,m(c-x)^{m-1}(-1) + (c-x)^m = 0$$

$$-mx(c-x)^{m-1} + (c-x)^m = 0$$

$$mx(c-x)^{m-1} = (c-x)^m$$

$$mx = \frac{(c-x)^m}{(c-x)^{m-1}}$$

$$mx = (c-x)$$

$$mx + x = c$$

$$x(m+1) = c$$

$$\boxed{x = \frac{c}{m+1}}$$

$\therefore f_{m+1}(c) = \max\limits_{0 \le x \le c} \left\{ x \left(\frac{c-x}{m} \right)^m \right\}$

$$= \left(\frac{c}{m+1}\right) \left(\frac{c - \frac{c}{m+1}}{m} \right)^m = \left(\frac{c}{m+1} \right)^{m+1}$$

$\therefore$ The result is also true for $n = m+1$

Hence by Mathematical induction, the optimal policy is $(c/n, c/n, \dots c/n)$ and $f_n(c) = (c/n)^n$.

(2) Solve the following by dynamic programming

$\min z = \underline{y_1 + y_2 + \dots + y_n}$.

Subject to constraints $\underline{\underline{y_1 \cdot y_2 \cdots y_n = b}}$

and $y_1, y_2, \dots y_n \ge 0$ (or) Factorize a positive quantity b into n factors in such a way so that their sum is a minimum.

__Solution:__ To develop the recursive ~~Problem~~ equation:

Let $f_n(b)$ be the minimum attainable sum $y_1 + y_2 + \dots + y_n$ when the positive quantity b is factorized into z factors $y_1, y_2, \dots y_n$.

For $n = 1$ (one stage problem)

Here b is factorized into one factor $y_1 = b$ only.

$\therefore f_1(b) = \max\limits_{y_1 = b} \{y_1\} = b$ (trivial case) $\longrightarrow (1)$

For $n = 2$ (Two stage problem)

Here b is factorized into one factor $y_1 = x$ and $y_2 = b/x$ so that $y_1 \cdot y_2 = b$.

Then $f_2(b) = \max\limits_{0 \le x \le b} \{y_1 + y_2\} = \max\limits_{0 \le x \le b} \{x + b/x\}$

$= \max\limits_{0 \le x \le b} \{x + f_1(b/x)\} \longrightarrow (2)$ like here ($\because f_1(b) = b$)

For $n = 3$ (Three stage problem)

Here b is factorized into three factors $y_1 = x$ and $y_2 y_3 = b/x$ so that $y_1 y_2 y_3 = b$.

i.e., b/x is further factorized into two factors whose minimum attainable sum is $f_2(b/x)$

Then $f_3(b) = \min\limits_{0 \le x \le b} \{y_1 + y_2 + y_3\}$

$= \min\limits_{0 \le x \le b} \{x + f_2(b/x)\}$ by (2)

In general, the recursive equation for the n-stage problem is $f_n(b) = \min\limits_{0 \le x \le b} \{x + f_{n-1}(b/x)\}$. $\longrightarrow (3)$

To solve the recursive equation.

For $n = 2$, eqn (3) becomes

$f_2(b) = \min\limits_{0 \le x \le b} \{x + f_1(b/x)\} = \min\limits_{0 \le x \le b} \{x + b/x\}$

The function $x + b/x$ will attain its minimum when $x = \sqrt{b}$.

$\frac{d}{dx}(x + b/x) = 0$ $\quad \frac{d}{dx}(1/x^n) = \frac{-n}{x^{n+1}}$

$1 - b/x^2 = 0$ $\quad \frac{-1}{x^{1+1}} = \frac{-1}{x^2}$

$b/x^2 = 1 \quad x^2 = b$

$1/x^2 = 1/b \quad x = \sqrt{b}$

$\therefore f_2(b) = \min_{0 \le x \le b} \{x + b/x\} = \sqrt{b} + \frac{b}{\sqrt{b}} = 2\sqrt{b} = 2(b)^{1/2}$

The optimal policy $(b^{1/2}, b^{1/2})$ is and $\underline{\underline{f_2(b) = 2b^{1/2}}}$

For $n = 3$, eqn (3) becomes

$$f_3(b) = \min_{0 \le x \le b} \{x + f_2(b/x)\} = \min_{0 \le x \le b} \left\{x + \underline{\underline{2(b/x)^{1/2}}}\right\}$$

The function $x + 2(b/x)^{1/2}$ will attain its minimum when $x = b^{1/3}$.

$$f_3(b) = \min_{0 \le x \le b} \left\{x + 2(b/x)^{1/2} = b^{1/3} + 2\left(\frac{b}{b^{1/3}}\right)^{1/2} = 3b^{1/3}\right\}$$

$\therefore$ The optimal policy is $(b^{1/3}, b^{1/3}, b^{1/3})$ and $f_3(b) = 3b^{1/3}$.

Let us assume that the optimal policy for $n = m$ is $(b^{1/m}, b^{1/m}, b^{1/m}, \ldots, b^{1/m})$ and $f_m(b) = m b^{1/m}$.

Now, for $n = m+1$, eqn (3) becomes

$$f_{m+1}(b) = \min_{0 \le x \le b} \{x + f_n(b/x)\} = \min_{0 \le x \le b} \{x + m(b/x)^{1/m}\}$$

The function $x + m(b/x)^{1/m}$ will attain its minimum when $x = b^{1/m+1}$ → back side

$$\therefore f_{m+1}(b) = \min_{0 \le x \le b} \{x + m(b/x)^{1/m}\}$$

$$= b^{1/m+1} + m\left(\frac{b}{b^{1/m+1}}\right)^{1/m+1} = (m+1)\, b^{\frac{1}{m+1}}$$

i.e., the result is also true for $n = m+1$.

Hence by Mathematical induction, the optimal policy is $(b^{1/n}, b^{1/n}, b^{1/n}, \ldots b^{1/n})$ and $f_n(b) = nb^{1/n}$

$$\frac{d}{dx}\left(x + 2(b/x)^{1/2}\right) = 0.$$

$$1 + \cancel{2} \cdot \tfrac{1}{\cancel{2}}(b/x)^{-1/2}(-b/x^2) = 0$$

$$1 - b/x^2 (b/x)^{-1/2} = 0.$$

$$1 - \frac{b/x^2}{(b/x)^{1/2}} = 0 \qquad 1 = \frac{b^{1/2}}{x^{3/2}}$$

$$\frac{(b/x)^{1/2} - (b/x^2)}{(b/x)^{1/2}} = 0 \qquad x^{3/2} = b^{1/2}$$

$$x^3 = b$$

$$\boxed{x = b^{1/3}}$$

$$\frac{b^{1/2}}{x^{1/2}} - \frac{b}{x^2} = 0$$

$$\frac{b^{1/2}}{x^{1/2}} = b/x^2 \Rightarrow \frac{1}{\cancel{x^{1/2}}}\,b^{1/2} = \frac{\cancel{b^{1/2}} \cdot b^{1/2}}{x^{1} \cdot x^{1/2} \ldots}$$