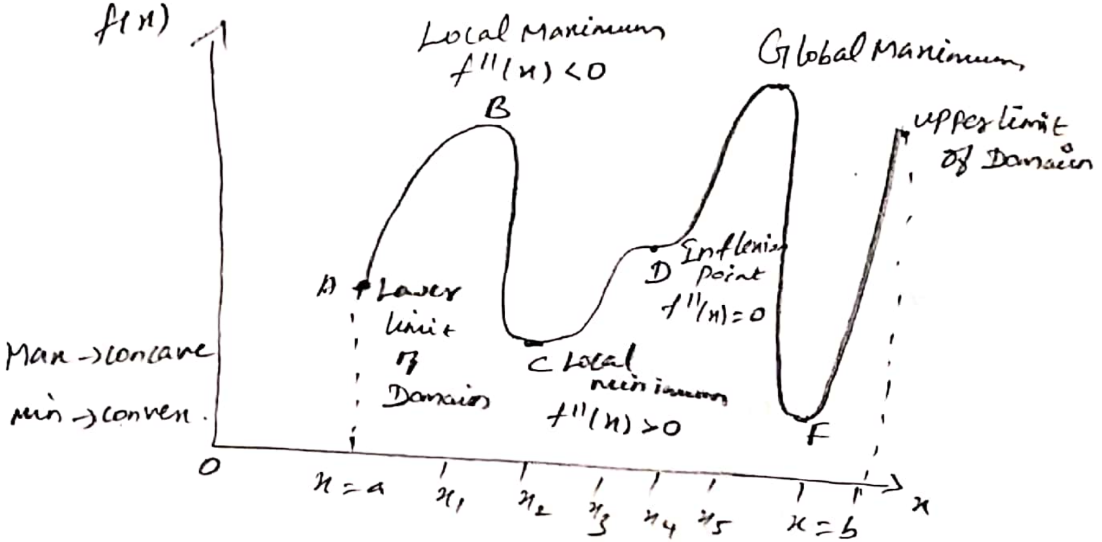
a) If n is odd, x_0 gives an inflexion point $(x_0, f(x_0))$

b) If n is even then x_0 makes $f(x_0)$ a minimum if $f^{(n)}(x_0) > 0$ and a Maximum if $f^{(n)}(x_0) < 0$.



Defn: Global minimum (or) Maximum

Global minimum (or Maximum) value of a function is the minimum (or Maximum) value among all local minimum (or Maximum) values of the function.

Necessary condition:

A necessary condition for a point x_0 to be the local extrema (local Maximum and minimum) of a function $y = f(x)$ defined in the interval $a \le x \le b$ is that the first derivative of $f(x)$ exists as a finite number at $x = x_0$ and $f'(x_0) = 0$.

concave ⇒ −ve convex ⇒ +ve

Unconstrained External Problems:

1. Find the Maxima or minima of the function
$$f(x) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$$

Sln: Given $f(x) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

We know that, the necessary condition is $\nabla f(x_0) = 0$.

$$\Rightarrow \frac{\partial f}{\partial x_1} = 0 \Rightarrow 1 - 2x_1 = 0 \rightarrow (1)$$

$$\Rightarrow \frac{\partial f}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0 \Rightarrow x_3 = 2x_2 \rightarrow (2)$$

$$\Rightarrow \frac{\partial f}{\partial x_3} = 0 \Rightarrow 2 + x_2 - 2x_3 = 0 \rightarrow (3)$$

Eqn (2) sub in eqn (3), we get.

$$2 + x_2 - 2(2x_2) = 0$$
$$2 - 3x_2 = 0$$
$$3x_2 = 2$$
$$\boxed{x_2 = 2/3}$$

From (2), $x_3 = 2(2/3) = 4/3 \Rightarrow \boxed{x_3 = 4/3}$

From (1), $2x_1 = 1 \Rightarrow \boxed{x_1 = 1/2}$

$\therefore X_0 = (x_1, x_2, x_3) = (1/2, 2/3, 4/3)$

To determine the type of the stationary point, consider

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

Here $\frac{\partial^2 f}{\partial x_1^2} = -2$, $\frac{\partial^2 f}{\partial x_2^2} = -2$, $\frac{\partial^2 f}{\partial x_3^2} = -2$

$\frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$, $\frac{\partial^2 f}{\partial x_2 \partial x_3} = 1$

$$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The principal minor determinants of H

$|-2|$, $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$, $\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix}$ are

-2, 4, -6

$\therefore$ The Hessian Matrix is negative definite and the ~~stationary~~ and hence, the function is concave $(-ve)$ $f''(x) < 0$. Stationary point $(1/2, 2/3, 4/3)$ is local Maximum,

$f(x)_{max} = 1/2 + 2(4/3) + (2/3)(4/3) - (1/2)^2 - (2/3)^2 - (4/3)^2$

$= \frac{57}{36} = \frac{19}{12}$

(2). Show that the functions whether it is convex, concave or neither: a) $f(x) = 10 - x^2$ b) $f(x) = x^4 + 6x^2 + 12x$ c) $f(x) = x^4 + x^2$

Soln: a) $f(x) = 10 - x^2$

$\therefore \frac{\partial f}{\partial x} = -2x$ and $\frac{\partial^2 f}{\partial x^2} = -2$

Since $\frac{\partial^2 f}{\partial x^2}$ is always < 0 for all values of x, the function is concave.

b) $f(x) = x^4 + 6x^2 + 12x$

$\frac{\partial f}{\partial x} = 4x^3 + 12x + 12$ and $\frac{\partial^2 f}{\partial x^2} = 12x^2 + 12$

Since $\frac{\partial^2 f}{\partial x^2} > 0$, for all values of x, the function is convex.

c) $f(x) = x^4 + x^2$

$\therefore \frac{\partial f}{\partial x} = 4x^3 + 2x$ & $\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2$

Since $\frac{\partial^2 f}{\partial x^2} > 0$ $\forall x$, the function is convex.

(3) Find the Stationary points of $f(x) = 4x^4 - x^2 + 5$ and determine the nature of the stationary points.

Solution: $f(x) = 4x^4 - x^2 + 5$

$f'(x) = 16x^3 - 2x$

$f'(x) = 0$

i.e., $16x^3 - 2x = 0$

$2x(8x^2 - 1) = 0$

$x = 0$ (or) $8x^2 - 1 = 0$

$x = 0$ (or) $8x^2 = 1$

$x^2 = 1/8$

$x = \pm \frac{1}{2\sqrt{2}}$

$\therefore$ The stationary points are $x = 0$, $x = \frac{1}{2\sqrt{2}}$, $x = \frac{-1}{2\sqrt{2}}$

To determine the nature of these values

$f''(x) = 48x^2 - 2$

Case(i): consider $x = 0$

$f''(0) = -2$ (-ve) max

$\therefore x = 0$ maximises $f(x)$ and the maximum value of is $f(x) = 5$

case(ii): consider $x = \frac{1}{2\sqrt{2}}$

$f''\left(\frac{1}{2\sqrt{2}}\right) = 48 \times \frac{1}{8} - 2 = 4 > 0$ (+ve) min

$x = \frac{1}{2\sqrt{2}}$ minimises $f(x)$ and the minimum value of $f(x)$ is $4 \times \frac{1}{64} - \frac{1}{8} + 5 = 79/16$

case(iii): consider $x = -\frac{1}{2\sqrt{2}}$

$f''\left(\frac{-1}{2\sqrt{2}}\right) = 48 \times \frac{1}{8} - 2 = 4 > 0$ +ve min

$\therefore x = -\frac{1}{2\sqrt{2}}$ also minimises $f(x)$ and the minimum value of $f(x) = 79/16$.

(4) Investigate $f(x) = x^4 - 2x^2 - 16x + 1$ for maxima and minima, use Newton-Raphson method to determine the extreme value correct to 3 decimal places.

Soln:

$f(x) = x^4 - 2x^2 - 16x + 1$; $f'(x) = 4x^3 - 4x - 16$

~~[$f(0) = 1 = +ve$~~ $f'(1) = 4 - 4 - 16 = -ve$

~~$f(1) = 1 - 2 - 16 + 1 = -16 = -ve$~~ $f'(2) = 32 - 8 - 16 = +ve$

~~$f(2) = 16 - 8 - 16 \times 2 + 1 = 16 - 40 + 1 = -ve$]~~

$f'(x) = 0 \Rightarrow 4x^3 - 4x - 16 = 0$

$x^3 - x - 4 = 0$

$f'(1) < 0$ and $f'(2) > 0$

$\therefore$ The roots lies between 1 and 2.

Take $x_0 = 1.5$

The Newton-Raphson formula is

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2 \ldots$

$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$

Put $n = 0$, where $f(x) = x^3 - x - 4$

Let $f'(x) = f''(x) = 3x^2$

$x_1 = \cancel{x_0 - \frac{f'(x_0)}{f''(x_0)}} = \cancel{1.5} \frac{2(1.5)^3 + (1.5) + 4}{3(1.5)^2 - 1} =$

$6.75 + 1.5$

$\frac{12.25}{5.75} = 2.130$

$$x_{k+1} = x_k - \frac{(x_k^3 - x_k - 4)}{3x_k^2 - 1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}$$

x_k	$\cancel{x_{k+1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}}$
x_0	1.5
x_1	1.869 ~~1.87~~
x_2	1.799 ~~1.80~~
x_3	1.796
x_4	1.796

$$x_1 = \frac{2x_0^3 + x_0 + 4}{3x_0^2 - 1} = \frac{2(1.5)^3 + 1.5 + 4}{3(1.5)^2 - 1}$$

The only extreme value is 1.796 correct to is 3 decimal places. Since two consecutive iterations agree at $k = 3, 4$

$$f''(x) = 12x^2 - 4$$

$$\therefore f''(1.796) = 12(1.796)^2 - 4 > 0$$

$$\therefore x = 1.796 \text{ minimizes } f(x)$$

$$\text{Min } f(x) = (1.796)^4 - 2(1.796)^2 - 16(1.796) + 1$$

$$= -23.783$$

$-3x_1 + 7x_2 = -3$

i.e., $3x_1 - 7x_2 = 3 \rightarrow$ (7)

From (6), $x_1 + x_2 = 6 \rightarrow$ (8)

Solve (7) & (8)

$$3x_1 - 7x_2 = 3$$

(8)×3 ⇒ $3x_1 + 3x_2 = 18$

$$\overline{-10x_2 = -15}$$

$$10x_2 = 15$$

$$x_2 = 3/2$$

$x_1 + 3/2 = 6 \Rightarrow x_1 = 6 - 3/2 = 9/2$

The solution is $x_1 = 9/2$, $x_2 = 3/2$.

$$\text{Max } z = 10(9/2) + 4(3/2) - \frac{81}{4} + 4 \times \frac{27}{4} - 5(9$$

$$= 78 - 63/2 = \frac{93}{2}$$

Now $n = 2$, $m = 1$, $n - m = 2 - 1 = 1$, principal minor of the following det of or

$$\therefore \Delta_{n+1} = \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -10 \end{vmatrix} = 20 > 0$$

$\therefore x_1 = 9/2$, $x_2 = 3/2$ Maximises $z = \frac{93}{2}$.

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right]$$

Constrained extremal Problem with more than one equality constraint.

(1) Solve the non-linear programming problem given below. Optimize $z = x_1^2 + x_2^2 + x_3^2$

Subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$.

Solution:

Lagrangian function can be written as

$$f(x, \lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The necessary conditions for the Maxima or minima are

$$\frac{\partial f}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0 \rightarrow (1)$$

$$\frac{\partial f}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0 \rightarrow (2)$$

$$\frac{\partial f}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0 \rightarrow (3)$$

$$\frac{\partial f}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0 \rightarrow (4)$$

and $\frac{\partial f}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \rightarrow (5)$

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From (1), $2x_1 = \lambda_1 + 5\lambda_2$

$$x_1 = \frac{\lambda_1 + 5\lambda_2}{2} \longrightarrow (6)$$

From (2), $2x_2 = \lambda_1 + 2\lambda_2$

$$x_2 = \frac{\lambda_1 + 2\lambda_2}{2} \longrightarrow (7)$$

From (3), $2x_3 = 3\lambda_1 + \lambda_2$

$$x_3 = \frac{3\lambda_1 + \lambda_2}{2} \longrightarrow (8)$$

Substituting the values of x_1, x_2, x_3 in eqn (4) & (5) we get.

eqn (4) $\Rightarrow x_1 + x_2 + 3x_3 - 2 = 0$.

$$\frac{\lambda_1 + 5\lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + 3\left(\frac{3\lambda_1 + \lambda_2}{2}\right) - 2 = 0$$

$$\lambda_1 + 5\lambda_2 + \lambda_1 + 2\lambda_2 + 9\lambda_1 + 3\lambda_2 - 4 = 0$$

$$11\lambda_1 + 10\lambda_2 = 4 \longrightarrow (9)$$

eqn (5) $\Rightarrow 5x_1 + 2x_2 + x_3 - 5 = 0$

$$5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) - 5 = 0$$

$$5\lambda_1 + 25\lambda_2 + 2\lambda_1 + 4\lambda_2 + 3\lambda_1 + \lambda_2 - 10 = 0$$

$$10\lambda_1 + 30\lambda_2 = 10$$

$$\lambda_1 + 3\lambda_2 = 1 \longrightarrow (10)$$

Solving eqn (9) & (10) we get.

$$11\lambda_1 + 10\lambda_2 = 4$$

eqn (10) $\times 11 \Rightarrow \underset{-}{11\lambda_1} \underset{-}{+33\lambda_2} = \underset{-}{11}$

$$-23\lambda_2 = -7$$

$$\lambda_2 = 7/23$$

$$\lambda_2 = 0.304$$

$\lambda_2 = 0.304$ sub in eqn (10) we get.

$$\lambda_1 + 3(0.304) = 1$$

$$\lambda_1 + 0.912 = 1 \Rightarrow \lambda_1 = 1 - 0.912$$

$\therefore \lambda_1 = 0.087$. $\lambda_1 = 2/23$

From (6), $x_1 = \frac{0.087 + 5(0.304)}{2}$

$x_1 = 0.804$ $\quad x_1 = 37/46$, $x_2 = 8/23$

From (7), $x_2 = \frac{(0.087) + 2(0.304)}{2} = 0.348$ $\quad x_3 = \frac{13}{46}$

From (8), $x_3 = \frac{3(0.087) + 0.304}{2} = 0.283$

as the solution.

To determine whether this solution point is a maxima or minima, the following Bordered Hessian Matrix is constructed:

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right]_{(m+n)\times(m+n)}$$

where $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

$P^T = \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\to x_1^2 + x_2^2 + x_3^2$ multiplied by 2

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right]$$

Since $n=3$, $m=2$, $n-m=1$ and $2m+1=5$. This shows that only one principal minor of H^B of order 5 needs to be solved.

For maximization, the sign should be $(-1)^{m+n} = (-1)^5 = -ve$ and for minimization, the sign should be $(-1)^m = (-1)^2 = +ve$. Now the determinant of H^B of order 5 is

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 0 & 2 & 1 \\ 1 & 5 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 & 5 & 1 \\ 1 & 5 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 & 5 & 2 \\ 1 & 5 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1\left[2\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix}\right]$$

$$-1\left[5\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}\right]$$

$$+3\left[5\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}\right]$$

$= \underline{460}$

Since, the value is +ve, the above solution minimizes the objective function and

$Z_{min} = (0.804)^2 + (0.348)^2 + (0.283)^2$

$= 0.847$.

Check.

$1\left[2\left[1(4-0)-5(2)\right]-1\left[1(-2)-5(-6)\right]\right.$

$-1\left[5\left(1(4)-5(2)\right)\right]-1\left[1(0)-5(0)+2(1-6)\right]$

$+3\left[5\left(1(-2)-5(-6)\right)\right]-2\left(1(0)-5(0)+2(1-6)\right]$

$= 1\left[2(4-10)-1(-2+30)\right]-1\left[5(4-10)-1(-10)\right]$

$+3\left[5(-2+30)-2(-10)\right]$

$= 1\left[2(-6)-1(28)\right]-1\left[5(-6)+10\right]$

$+3\left[5(28)+20\right]$

$= -12-28+30-10+3 \cancel{4} 420+60$.

$= 510 - 50$ — 510, 50

$= 460 = 11$ correct // Answer

Kuhn-Tucker condition:

One-inequality constraint

(1) Maximise $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Subject to $3x_1 + 2x_2 \le 6$, $x_1, x_2 \ge 0$.

Solution:

Let $f(x) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$h(x) = 3x_1 + 2x_2 - 6$

Kuhn-Tucker conditions for the maximisation problem become

$\frac{\partial f(x)}{\partial x_j} - \lambda \frac{\partial h(x)}{\partial x_j} = 0$ (or) $f(x) - \lambda h(x) = 0$, $j = 1, 2$

$\lambda h(x) = 0$, $\lambda \ge 0$.

$h(x) \le 0$

$8x_1 + 10x_2 - x_1^2 - x_2^2 - \lambda(3x_1 + 2x_2 - 6) = 0$

p.d. w.r. to x_1 we get.

$8 - 2x_1 - 3\lambda = 0 \longrightarrow (1)$

p.d. w.r. to x_2 we get

$10 - 2x_2 - 2\lambda = 0 \longrightarrow (2)$

p.d. w.r. to λ, we get.

$(3x_1 + 2x_2 - 6) = 0 \longrightarrow (3)$

$(3x_1 + 2x_2 - 6) \le 0$ and $\lambda \ge 0 \longrightarrow (4)$

eqn (3) $\Rightarrow \lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$

Case(i): $\lambda = 0$

The above equations become

$8 - 2x_1 = 0 \Rightarrow x_1 = 4$

$10 - 2x_2 = 0 \Rightarrow x_2 = 5$

$\therefore x_1 = 4,\ x_2 = 5$

This solution is not feasible. Since $3x_1 + 2x_2 \neq 6$ when $x_1 = 4$ and $x_2 = 5$

$12 + 10 = 6$
$22 \neq 6$

Case(ii): $\lambda \neq 0$

From (1), $x_1 = \frac{8-3\lambda}{2}$

From (2), $x_2 = \frac{10-2\lambda}{2} = 5 - \lambda$

The values of x_1, x_2 sub in eqn (3) we get

$3\left(\frac{8-3\lambda}{2}\right) + 2(5-\lambda) - 6 = 0$

$24 - 9\lambda + 20 - 4\lambda - 12 = 0$

$44 - 12 - 13\lambda = 0$

$32 - 13\lambda = 0$

$13\lambda - 32 = 0$

$13\lambda = 32$

$\boxed{\lambda = \frac{32}{13}}$

$x_1 = \frac{8-3\lambda}{2} = \frac{8 - 3\left(\frac{32}{13}\right)}{2} = 4 - 3\left(\frac{16}{13}\right) = \frac{4}{13}$

$4 - 3\left(\frac{16}{13}\right)$, $\frac{52-48}{13}$, $\frac{52-48}{13} = 4/13$

$x_2 = \frac{10-2\lambda}{2} = \frac{10 - 2\left(\frac{32}{13}\right)}{2} = 5 - \frac{32}{13} = \frac{33}{13}$

$\frac{65-32}{13}$

$3x_1 + 2x_2 = 6$

$\therefore x_1 = 4/13,\ x_2 = \frac{33}{13}$.

$\therefore$ This solution is feasible, since $3x_1 + 2x_2 = 6$ when $x_1 = 4/13$, $x_2 = \frac{33}{13}$

$3(4/13) + 2\left(\frac{33}{13}\right) = 6$

$\frac{12}{13} + \frac{66}{13} = 6$

$\frac{78}{13} = 6$

$6 = 6$

$\text{Max } z = f(x_1, x_2) = 8(4/13) + 10\left(\frac{33}{13}\right) - (4/13)^2 - \left(\frac{33}{13}\right)^2$

$= \frac{32}{13} + \frac{330}{13} - \frac{16}{169} - \frac{1089}{169}$

$= \frac{362}{13} - \frac{1105}{169}$

$= 27.84 - 6.5 = 21.3$

$= \frac{3601}{169}$

$\therefore \text{Max } z = 21.3$

$\therefore$ The solution is $x_1 = 4/13$, $x_2 = \frac{33}{13}$

$\text{Max } z = 21.3$

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Equality constraints : Problems

Jacobian method

(1) Minimize $f(x) = x_1^2 + x_2^2 + x_3^2$

Subject to $g_1(X) = x_1 + x_2 + 3x_3 - 2 = 0$

$g_2(X) = 5x_1 + 2x_2 + x_3 - 5 = 0$

Soln: To find the constrained extreme points as follows. Let $Y = (x_1, x_2)$ and $X = x_3$

$$\nabla_Y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (2x_1, 2x_2)$$

$$\nabla_X f = \frac{\partial f}{\partial x_3} = 2x_3$$

$|J| = 2-5 = -3$

$\text{adj}\, J = \begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix}$ replace

$$J = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix}, \quad J^{-1} = \frac{1}{|J|}\,\text{adj}\, J$$

$$J^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \nabla_c f = \frac{\partial_c f}{\partial_c x_3} = 2x_3 - (2x_1, 2x_2)\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= 2x_3 - (2x_1, 2x_2)\begin{pmatrix} -\frac{6}{3} + \frac{1}{3} \\ \frac{15}{3} - \frac{1}{3} \end{pmatrix}$$

$$= 2x_3 - (2x_1, 2x_2)\begin{pmatrix} -\frac{5}{3} \\ \frac{14}{3} \end{pmatrix}$$

$$= 2x_3 - \left(-\frac{10x_1}{3} + \frac{28x_2}{3}\right) \quad J^{-1}c$$

$$= \frac{10x_1}{3} - \frac{28}{3}x_2 + 2x_3$$

The equations for determining the stationary points are given as

$\nabla_c f = 0$.

$g_1(x) = 0$ (or) $\begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$,

$g_2(x) = 0$

$10x_1 - 28x_2 + 6x_3 = 0$

$x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

we get

Hence $x_0 \approx \tilde{x}$ $(0.81, 0.35, 0.28)$

The identity of this stationary point is verified using the sufficiency condition.

Given that x_3 is the independent variable, it follows from $\nabla_c f$ that

$$\frac{\partial_c^2 f}{\partial_c x_3^2} = \frac{10}{3}\left(\frac{dx_1}{dx_3}\right) - \frac{28}{3}\left(\frac{dx_2}{dx_3}\right) + 2$$

$$= \begin{pmatrix} \frac{10}{3} & -\frac{28}{3} \end{pmatrix} \begin{pmatrix} \frac{dx_1}{dx_3} \\ \frac{dx_2}{dx_3} \end{pmatrix} + 2$$

From the Jacobian method

$$\begin{pmatrix} \frac{dx_1}{dx_3} \\ \frac{dx_2}{dx_3} \end{pmatrix} = -J^{-1}C = -\begin{pmatrix} -5/3 \\ 14/3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ -14/3 \end{pmatrix}$$

$$\therefore \frac{\partial_c^2 f}{\partial_c x_3^2} = \frac{10}{3}\left(\frac{5}{3}\right) - \frac{28}{3}\left(-\frac{14}{3}\right) + \frac{2 \times 9}{9} \text{ L.C.M}$$

$\therefore \frac{\partial_c^2 f}{\partial_c x_3^2} = \frac{460}{9} > 0$ Hence X_0 is the minimum point

More than one inequality constraint

① Maximise $z = 7x_1^2 + 6x_1 + 5x_2$

Subject to $x_1 + 2x_2 \le 10$

$x_1 - 3x_2 \le 9$, $x_1, x_2 \ge 0$.

Solution:

Let $f(x) = 7x_1^2 + 6x_1 + 5x_2$

$h^1(x) = x_1 + 2x_2 - 10$

$h^2(x) = x_1 - 3x_2 - 9$

The Kuhn–Tucker conditions for the maximization problem become

$f_j(x) - \sum_{i=1}^{2} \lambda_i h^i(x) = 0$ $\quad i = 1, 2$.

$\lambda_1 h^1(x) = 0$ $\quad j = 1, 2$,

$\lambda_2 h^2(x) = 0$

$h^i(x) \le 0$ $\quad i = 1, 2$.

$\lambda_1, \lambda_2 \ge 0$. $\quad x_1, x_2 \ge 0$.

$7x_1^2 + 6x_1 + 5x_2 - \lambda_1(x_1 + 2x_2 - 10) - \lambda_2(x_1 - 3x_2 - 9) = 0$

P.D. w.r.to x_1 we get .

$14x_1 + 6 - \lambda_1 - \lambda_2 = 0 \longrightarrow$ ①

P.D. w.r.to x_2 we get

$5 - 2\lambda_1 + 3\lambda_2 = 0 \longrightarrow$ ②

$\lambda_1(x_1+2x_2-10)=0 \rightarrow (3)$

$\lambda_2(x_1-3x_2-9)=0 \rightarrow (4)$

$x_1+2x_2-10 \le 0 \rightarrow (5)$

$x_1-3x_2-9 \le 0 \rightarrow (6)$

$x_1, x_2 \ge 0 \quad , \lambda_1, \lambda_2 \ge 0.$

Case (i):

$\lambda_1=0, \quad \lambda_2=0$

Equation (2) gives absurd result namely $5=0$. No feasible solution in this case.

Case (ii): $\lambda_1=0, \quad \lambda_2 \ne 0.$

eqn (1) becomes $14x_1+6-\lambda_2=0$

$\lambda_2=14x_1+6.$

$\cancel{x_1-3x_2-9}=0.$

$5+3\lambda_2=0$

$5+3(14x_1+6)=0.$

$5+42x_1+18=0$

$42x_1=-23$

$x_1=\frac{-23}{42}<0$

Case (iii): $\lambda_1 \ne 0, \quad \lambda_2=0$

$14x_1+6-\lambda_1=0$

$5-2\lambda_1=0$

$\cancel{x_1+2x_2-10}=0$

$\lambda_1=14x_1+6$

$\therefore 5-2(14x_1+6)=0$

$5-28x_1-12=0$

$-28x_1-7=0$

$x_1=\frac{-\cancel{7}\,1}{\cancel{28}\,4}<0$

No feasible solution in this case.

Case (iv): $\lambda_1 \ne 0, \quad \lambda_2 \ne 0.$

$\cancel{x_1}+2x_2-10=0$

$\cancel{x_1}-3x_2-9=0$

$\quad - \quad + \quad +$

$5x_2-1=0,$

$5x_2=1$

$\boxed{x_2=1/5}$

$x_1+2x_2-10=0$

$\cancel{2x_2=10-1/5}=49/5$

$x_2=\frac{49}{5\times2}$